PERIODS

In this talk, I'd like to tell you a few things about numbers. I'll assume that, by some divine power, we have been granted the ring $\mathbf{Z}$ of integers[1].

Definition 1. Let $\mathbf{C}$ be the algebraic closure of the completion $\mathbf{R}$ of $\mathbf{Q} = \mathbf{Z}[\frac{1}{p},\ p \text{ prime}]$ at the infinite place. A *number* is an element of $\mathbf{C}$.

There are far too many numbers, so we want to study a special class of such. According to our definition of $\mathbf{C}$, it's built from $\mathbf{R}$, but even this has far too many elements. Since $\mathbf{R}$ is itself built from $\mathbf{Q}$, which is at least countable, let us declare that the nicest numbers of all are elements of $\mathbf{Q}$.

With our 20,000 years of experience trying to find better ways to count, we've found some rules which allow us to play with numbers. For example, we can construct polynomials from previously constructed numbers, and try to solve them.

Definition 2. An *algebraic number* is an element of $\mathbf{C}$ which can be written as a root of a nonzero polynomial with coefficients in $\mathbf{Q}$; in other words, an algebraic number is an element of the algebraic closure $\overline{\mathbf{Q}}$. A *transcendental number* is an element of the complement $\mathbf{C} \smallsetminus \overline{\mathbf{Q}}$.

Algebraic numbers are really nice, and include numbers like $\sqrt{2}$ and i. But there are a lot of numbers which we've found control our lives that are transcendental:

Theorem 3 (Lindemann)**.** *If $x \in \overline{\mathbf{Q}}$, then $e^x \in \mathbf{C} \smallsetminus \overline{\mathbf{Q}}$.*

As a consequence, if $x \in \mathbf{Q}$, then $\log(x) \in \mathbf{C} \smallsetminus \overline{\mathbf{Q}}$. Similarly, π is transcendental (if not, then $i\pi$ would be an algebraic number; but then Lindemann tells us that $e^{i\pi} = -1$ would be transcendental). I like to think about characteristic p a lot, which means that the exponential is a vicious untameable beast. So let's just say that e is still a scary number, and that I'd like to gain some more confidence before trying to call it a "nice" number. But π is very nice; even if I didn't believe that, I'd have no choice but to get used to it if I wished to understand geometry.

Therefore, I'd like to expand my notion of "niceness" to include some transcendental numbers (like π and $\log(2)$) as well. There are many ways in which one can try to do this: for instance, there are series expansions of π and $\log(2)$ (and e) given by

$$\pi = 4\tan^{-1}(1) = \sum_{n\geq 1} \frac{(-1)^{n+1}}{2n-1},\ \log(2) = \sum_{n\geq 1} \frac{(-1)^{n+1}}{n},\ e = \sum_{n\geq 1} \frac{1}{n!}.$$

Perhaps, then, we can try to define a subset of $\mathbf{C}$ consisting of numbers which can be expressed as a well-behaved infinite sum where each term has a consistent expression as a fraction $\frac{f}{g}$ of some functions f, g. This phrase is very suggestive of thinking about *integrals.* Moreover, it suggests a way to exclude e from the list of "nice" numbers: if we ask that f and g be *polynomials*, then we exclude $\frac{1}{n!}$.

Let us therefore make a definition, following Kontsevich and Zagier:

Definition 4. An element $a \in \mathbf{R}$ is called a *period* if it can be written as an absolutely convergent integral $\int_\Delta \frac{f(x_1,\cdots,x_n)}{g(x_1,\cdots,x_n)} dx_1 \cdots dx_n$, where f and g are multivariable polynomials with coefficients in

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[1] In some other parallel universe, we may have been given $\mathbf{F}_p[t]$ instead. I think Deligne is a being from that universe.

$\mathbf{Q}$, and $\Delta \subseteq \mathbf{R}^n$ is defined by polynomial inequalities with rational coefficients. An element $a \in \mathbf{C}$ is called a *period* if its real and imaginary parts are periods.

One can show that periods are closed under addition and multiplication, so they form a subring $\mathcal{P} \subseteq \mathbf{C}$. Let us see some examples.

Example 5. The number π is a period, since

$$\pi = \int_{x^2+y^2 \leq 1} dxdy = \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \cdots$$

Note that there can be many different integral representations for a period.

Example 6. Logarithms are periods, since $\log(a) = \int_1^a \frac{dx}{x}$.

Example 7. Let $n \geq 2$ be an integer. Then $\zeta(n) = \sum_{d\geq 1} \frac{1}{d^n}$ is a period. Indeed,

$$\zeta(n) = \int_{0<x_n<\cdots<x_1<1} \frac{dx_1}{x_1} \cdots \frac{dx_{n-1}}{x_{n-1}} \frac{dx_n}{1-x_n}.$$

For instance,

$$\zeta(2) = \int_0^1 \int_0^y \frac{dx}{1-x} \frac{dy}{y} = \int_0^1 \int_0^y \sum_{d\geq 1} x^{d-1} dxdy = \sum_{d\geq 1} \frac{1}{d} \int_0^1 y^{n-1} dy = \sum_{d\geq 1} \frac{1}{d^2}.$$

We will try to explore periods in some detail today. Observe that there are some natural operations defined on integrals of the form $\int_\Delta \frac{f(x_1,\cdots,x_n)}{g(x_1,\cdots,x_n)} dx_1 \cdots dx_n$: we can

- Add the integrands, or take unions of integration domains.
- Change variables by an invertible transformation $\mathbf{R}^n \to \mathbf{R}^n$.
- Use the fundamental theorem of calculus/Stokes' theorem to relate $\int_\Delta \frac{f}{g} d^n x$ to $\int_{\partial\Delta} \frac{a}{b} d^{n-1} x$ for some functions a, b, f, g.

These operations allow us to conjure a new integral representation of a period from a given integral representation.

Conjecture 8 (Kontsevich-Zagier)**.** *One can pass between* any *two integral representations of a period by one of the above three rules.*

Let us see this conjecture in action.

Proposition 9 (Calabi)**.** *Euler's equality $\zeta(2) = \frac{\pi^2}{6}$ can be obtained by using the above three rules to some integral representations of $\zeta(2)$ and $\frac{\pi^2}{6}$.*

Proof. Let us write $\zeta(2) = \frac{1}{3} \sum_{n\geq 0} \frac{4}{(2n+1)^2}$. Then, by using the geometric series expansion, one can show that

$$\sum_{n\geq 0} \frac{4}{(2n+1)^2} = \int_{(0,1)^2} \frac{1}{1-xy} \frac{dx}{\sqrt{x}} \frac{dy}{\sqrt{y}}.$$

Let us call this integral I; we wish to show that $I = \frac{\pi^2}{2}$ using only the above three rules. We then consider the change of variables given by the expressions

$$x = \frac{z^2(1+w^2)}{1+z^2},\ y = \frac{w^2(1+z^2)}{1+w^2}.$$

The Jacobian of this coordinate change is $4\frac{(1-xy)\sqrt{xy}}{(1+w^2)(1+z^2)}$, and the integration domain is changed to $z, w \geq 0,\ zw \leq 1$. Therefore,

$$I = 4 \int_{z,w\geq 0, zw \leq 1} \frac{dw}{1+w^2} \frac{dz}{1+z^2} = 2 \int_{z,w\geq 0} \frac{dw}{1+w^2} \frac{dz}{1+z^2} = 2 \left(\int_{z\geq 0} \frac{dz}{1+z^2} \right)^2,$$

where the final equality requires some manipulation (using $(z,w)\mapsto(z^{-1},w^{-1})$). We know that the integral inside the parentheses is $\pi/2$, so we see that $I=\frac{\pi^2}{2}$ only using the above three rules. □

The geometers' spidey-senses were probably tingling when we were listing the three rules above (and perhaps when defining periods): one might expect that there is a more abstract definition of periods, perhaps involving smooth manifolds and integration of differential forms along smooth chains. It's at this point that the talk (unfortunately) leaves the realm of what has so far (perhaps) been accessible to undergraduates.

Definition 10. Let us redefine a period as a number α for which there exists a d-dimensional smooth algebraic variety X over $\mathbf{Q}$, a divisor $D\subseteq X$ with normal crossings, a global algebraic d-form $\omega\in\Omega_X^d$ (so both ω and D are defined over $\mathbf{Q}$), and a singular relative chain γ on $X(\mathbf{C})$ (so the boundary of γ is in $D(\mathbf{C})$), such that $\alpha=\int_\gamma\omega$. Let $\mathcal{P}'$ be the subring of $\mathbf{C}$ consisting of this notion of periods.

Example 11. Let α be an algebraic number which is a root of a polynomial $f(x)\in\mathbf{Q}[x]$. Then α is the integral $\int_0^\alpha dx$ with $X=\mathbf{A}^1$, $D=V(xf)$, $\omega=dx$, and γ being the path from 0 to α in $\mathbf{A}^1(\mathbf{C})=\mathbf{C}$.

Example 12. Let $X=\mathbf{G}_m$, $D=\varnothing$, $\omega=\frac{dz}{z}$, and γ be the path $t\mapsto e^{2\pi it}$ in $\mathbf{G}_m(\mathbf{C})=\mathbf{C}\smallsetminus\{0\}$. Then $\int_\gamma\omega=2\pi i$.

Theorem 13 (Huber–Müller-Stach)**.** *There is an equality $\mathcal{P}=\mathcal{P}'$ of subrings of $\mathbf{C}$. In other words, the sophisticated and "elementary" definitions of periods agree.*

The above definition of periods is still not satisfactory, since we have not singled out the word "integrating over cycles" (although we have at least uttered the word "cycle"). As we know from algebraic topology, the most natural way to define integration is by passing to cohomology. This motivates:

Definition 14. Let us again redefine a period as a number α for which there exists a smooth algebraic variety X over $\mathbf{Q}$ and a subvariety $D\subseteq X$ such that α is in the image of the pairing

$$\int:\mathrm{H}^i(X(\mathbf{C}),D(\mathbf{C});\mathbf{Q})\otimes\mathrm{H}^i_{\mathrm{dR}}(X,D)\to\mathbf{C}.$$

Note that $\mathrm{H}^*_{\mathrm{dR}}$ denotes the (relative) *algebraic* de Rham cohomology (so it is defined via algebraic forms on X rel D which are defined over $\mathbf{Q}$). Alternatively, α is a matrix coefficient of the *period isomorphism*

$$\mathrm{H}^i_{\mathrm{dR}}(X,D)\otimes_{\mathbf{Q}}\mathbf{C}\xrightarrow{\sim}\mathrm{H}^*(X(\mathbf{C}),D(\mathbf{C}))\otimes_{\mathbf{Q}}\mathbf{C}$$

for some choice of $\mathbf{Q}$-bases of both sides. Let $\mathcal{P}''$ denote the subring of $\mathbf{C}$ consisting of this further redefinition of periods.

Theorem 15 (Huber–Müller-Stach)**.** *There is an equality $\mathcal{P}=\mathcal{P}'=\mathcal{P}''$ of subrings of $\mathbf{C}$. In other words, all our definitions of periods agree.*

It is now natural to ask: can we rephrase the Kontsevich-Zagier conjecture in this way? The answer is yes, but it is not a very deep rephrasing.

Definition 16. Let $\mathcal{P}_{\mathrm{KZ}}$ denote the ring generated by formal symbols (X,D,n,ω,γ), where X is a smooth algebraic variety over $\mathbf{Q}$, $D\subseteq X$ is a closed subvariety, $n\geq 0$ is an integer, $\omega\in\mathrm{H}^n_{\mathrm{dR}}(X,D)$, and $\gamma\in\mathrm{H}_i(X(\mathbf{C}),D(\mathbf{C});\mathbf{Q})$. We will simply denote such a tuple by $\int_\gamma\omega$, and subject these tuples to the relations

$$\int_{\gamma+\gamma'} \omega = \int_\gamma \omega + \int_{\gamma'} \omega,$$
$$\int_\gamma \omega + \omega' = \int_\gamma \omega + \int_\gamma \omega',$$
$$\int_{f_*\gamma} \omega = \int_\gamma f^*\omega,$$
$$\int_{\partial\gamma} \omega = \int_\gamma d\omega,$$

where the reader is left to unravel the meaning of the symbols.

Note that $\mathcal{P}_{\mathrm{KZ}}$ is only a ring, and *not* a field.

Example 17. Let ξ denote the element of $\mathcal{P}_{\mathrm{KZ}}$ defined by the data $X = \mathbf{G}_m$, $D = \varnothing$, $\omega = \frac{dz}{z}$, and γ being the path $t \mapsto e^{2\pi i t}$ in $\mathbf{G}_m(\mathbf{C}) = \mathbf{C} \smallsetminus \{0\}$. In the motivic literature, this is sometimes called the *Tate motive*.

Conjecture 18 (Rephrasing of Kontsevich-Zagier)**.** *The map* $\mathcal{P}_{\mathrm{KZ}}[\frac{1}{\xi}] \to \mathcal{P}$ *sending* $(X, D, n, \omega, \gamma) \mapsto \int_\gamma \omega$ *is an isomorphism.*

At this point, there are two directions in which this talk could go. Either we could talk about the motivic point of view on periods, or talk about *exponential periods*. Since I complained that the exponential scares me, I should face my fears and talk about it instead of attempting the Herculean task of explaining motives (in what I expect will be ten minutes of remaining time). Let us follow the same recipe as before: first, we will give the "elementary" definition of exponential periods, and then give a more sophisticated definition using de Rham cohomology. This will still not explain *why* the exponential has a right to exist, but I was not expecting to give a coherent answer to that anyway.

Definition 19. An element $a \in \mathbf{R}$ is called an *exponential period* if it can be written as an absolutely convergent integral $\int_\Delta e^{-V(x_1,\cdots,x_n)} \frac{f(x_1,\cdots,x_n)}{g(x_1,\cdots,x_n)} dx_1 \cdots dx_n$, where f, g, and V are multivariable polynomials with coefficients in $\mathbf{Q}$, and $\Delta \subseteq \mathbf{R}^n$ is defined by polynomial inequalities with rational coefficients. An element $a \in \mathbf{C}$ is called an *exponential period* if its real and imaginary parts are periods. Let $\mathcal{P}_{\mathrm{exp}}$ denote the subring of $\mathbf{C}$ consisting of exponential periods.

The notion of exponential periods includes many other constants of classical interest:

Example 20. The number $\sqrt{\pi}$ is an exponential period (and it is believed that $\sqrt{\pi}$ is *not* a period). Indeed, this follows from the famous calculation

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

This is perhaps the most important conceptual example of an exponential period. Indeed, note that $\sqrt{2\pi i}$ is an exponential period (since $\sqrt{i} = \frac{1+i}{\sqrt{2}}$ and $\sqrt{2}$ are both periods). It is the *square root* of the period $2\pi i$ from Example 12. From the motivic point of view, the period $2\pi i$ is absolutely fundamental (it corresponds to the element $\xi \in \mathcal{P}_{\mathrm{KZ}}$). Therefore, exponential periods should perhaps be viewed as obtained from a *square root* of ξ. (My belief is that this is *why* the exponential exists.) One can show that (in general) there is no square root of the motive of $\mathbf{G}_m$, which is evidence for the belief that $\sqrt{\pi}$ cannot be a period in the usual sense.

Example 21. Let γ be the Euler-Mascheroni constant, so

$$\gamma = \lim_{n\to\infty} \left(-\ln(n) + \sum_{d=1}^{n} \frac{1}{d} \right) = \int_1^\infty \left(\frac{1}{\lfloor x \rfloor} - \frac{1}{x} \right) dx = -\int_0^\infty e^{-x} \log(x) dx,$$

where the final integral representation is nontrivial to derive. Kontsevich conjectured that γ is an exponential period, and it was observed shortly after that since $\log(x) = \int_1^x \frac{dy}{dx}$, we can rewrite the final integral above as

$$\gamma = -\int_0^\infty \int_1^x \frac{e^{-x}}{y} dy dx,$$

thereby showing $\gamma \in \mathcal{P}_{\text{exp}}$.

Let us end by giving the definition of $\mathcal{P}_{\text{exp}}$ via de Rham cohomology:

Definition 22. Let us redefine an exponential period as a number α for which there exists a smooth algebraic variety X over $\mathbf{Q}$, a regular function $f : X \to \mathbf{A}^1$, and a subvariety $D \subseteq X$ such that α is in the image of the pairing

$$\int : \mathrm{H}^i(X(\mathbf{C}), D(\mathbf{C}), f; \mathbf{Q}) \otimes \mathrm{H}^i_{\mathrm{dR}}(X, D, f) \to \mathbf{C}.$$

Here, $\mathrm{H}^*_{\mathrm{dR}}(X, D, f)$ is defined by a "logarithmic" de Rham complex with twisted differential. If $D = \varnothing$, for instance, $\mathrm{H}^*_{\mathrm{dR}}(X, f)$ is defined as the hypercohomology of the complex

$$0 \to \mathcal{O}_X \xrightarrow{d+df} \Omega^1_X \xrightarrow{d+df} \Omega^2_X \to \cdots$$

Similarly, $\mathrm{H}^i(X(\mathbf{C}), D(\mathbf{C}), f; \mathbf{Q})$ is abusive notation for a modification of singular cohomology, defined (roughly) via smooth chains γ on $X(\mathbf{C})$ with boundary in $D(\mathbf{C})$, such that e^{-f} is of rapid decay along γ (so integrating e^{-f} along γ behaves well). Let $\mathcal{P}''_{\text{exp}}$ denote the subring of $\mathbf{C}$ consisting of this redefinition of exponential periods.

I think that $\mathcal{P}_{\text{exp}} = \mathcal{P}''_{\text{exp}}$, but I couldn't find a reference. My personal takeaway from this discussion (esp. Example 20) is that by considering square roots of the fundamental element/Tate motive $\xi \in \mathcal{P}_{\text{KZ}}$, which in turn is naturally related to replacing the study of algebraic varieties X with the study of *pairs* (X, f) where $f : X \to \mathbf{A}^1$ is a regular function on X, one naturally ends up in the realm of exponential periods.

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