Real Stable Polynomials: Description and Application

Theo McKenzie
theomckenzie@college.harvard.edu (347) 351-4092
Advisor: Peter Csikvari

March 23, 2015

Abstract

In this paper, we discuss the concept of stable polynomials. We go through some properties of these polynomials and then two applications: Gurvits' proof of the van der Waerden Conjecture and a proof that there exists an infinite family of d–regular bipartite Ramanujan graphs.

Contents

1 Real Stable Polynomials, and basic properties

In this paper we will discuss the concept and use of real stable polynomials, a seemingly simple concept that has lead to complex and involved results. Define the subspaces $z \in \mathbb{C}_+$ if $\mathrm{Re}(z) \geq 0$ and $z \in \mathbb{C}_{++}$ if $\mathrm{Re}(z) > 0$. Also $\mathbb{C}_+^n = \{(z_1, \dots, z_n) : z_i \in \mathbb{C}_+, 1 \leq i \leq n\}$ and $\mathbb{C}_{++}^n = \{(z_1, \dots, z_n) : z_i \in \mathbb{C}_{++}, 1 \leq i \leq n\}$. An n variable polynomial $p(z_1, \dots, z_n)$ is called *real stable* if it has real coefficients and all of its roots lie in the closed left half plane. Namely, we say that $p(z_1, \ldots, z_n)$ is real stable if $p(z_1, \ldots, z_n) \neq 0$ for all $(z_1, \ldots, z_n) \in \mathbb{C}^n_{++}$.

The above definition of stability is in fact only one of many definitions found in mathematical literature. Generally, a stable polynomial can refer to any polynomial that does not have roots in a defined region of the complex plane. The single variable version of our definition is called a Hurwitz polynomial. Another type is called *Schur polynomials*, which are multivariable polynomials that contain all of their roots in the open unit disk, for example $4z^3+3z^2+2z+1$. Also of some interest are polynomials that have no roots in the upper half of the complex plane (we will be looking at these in Section 3).

It also should be noted that it is easy to transform a polynomial under one definition of stability into that of another, as we can use a transformation to map the defined subset with no roots onto the other. For example we can test to see of a polynomial $P(x)$ of degree d is a single variable Schur polynomial by examining the polynomial:

$$Q(z) = (z-1)^d P\left(\frac{z+1}{z-1}\right).$$

If $Q(z)$ is Hurwitz stable, then $P(z)$ is a Schur polynomial, as the Möbius transformation $z \to \frac{z+1}{z-1}$ maps the unit disk to the right half plane.

1.1 Examples

A basic example of a real stable polynomial is

$$x^2 + 4x + 4$$

which factors to $(x+2)^2$.

A multivariate example of a stable polynomial is

$$1 + xy.$$

Call $x = r_1 e^{i\theta}$ and $y = r_2 e^{i\phi}$ for $r_1, r_2 > 0$ and $-\pi < \theta, \phi \leq \pi$. In order to be a root of this polynomial $\theta + \phi = \pi$ or $\theta + \phi = -\pi$. However for $e^{i\psi} \in \mathbb{C}_{++}$ it is the case that $-\frac{\pi}{2} < \psi < \frac{\pi}{2}$, meaning that there is no solution to $1 + xy$ in $\mathbb{C}^2_{++}$, therefore this polynomial is real stable.

A more complicated example is that of a Kirchhoff Polynomial, which is strongly related to a method of finding the number of spanning trees of a graph in polynomial time. Let $G = (V, E)$ be a connected graph with vertex set V and edges E. Each edge e is given a variable x_e. The Kirchhoff polynomial is

$$\mathrm{Kir}(G, \mathbf{x}) = \sum_T \prod_e x_e$$

where $\mathbf{x}$ is the set of variables x_e and T signifies the set of spanning trees of G.

As an example of the Kirchoff polynomial, take the graph in Figure 1 with the given variables. The Kirchhoff polynomial is

$$x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_5 + x_1x_4x_5 + x_2x_3x_4 + x_2x_3x_5 + x_2x_4x_5 + x_3x_4x_5$$

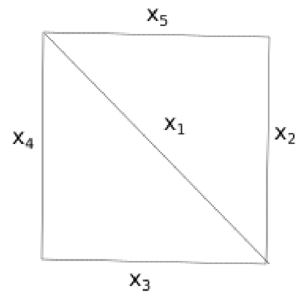

Figure 1: An example of the Kirchhoff polynomial

Proposition 1.1. *For a connected graph G, the Kirchhoff Polynomial Kir(G,$\boldsymbol{x}$) is real stable.*

We assume that G is connected considering that otherwise $\mathrm{Kir}(G, \mathbf{x}) = 0$.

For a graph $G = (V, E)$ call $n = |V|$ the number of vertices of G. We number the vertices $v_1, \ldots, v_n$ and give each edge $e \in E$ the variable x_e. We define the $n \times n$ *adjacency matrix* $A(\mathbf{x})$, such that $a_{i,j} = x_e$ if and only if there is an edge between v_i and v_j with variable x_e. Otherwise $a_{i,j} = 0$.

We define the *diagonal matrix* $D(\mathbf{x})$ as an $n \times n$ matrix such that $a_{i,i} = \sum x_f$ for $f \in F$ where F is the set of edges adjacent to the vertex v_i. If $i \neq j$ then $a_{i,j} = 0$.

We define the *Laplacian matrix* $L(\mathbf{x}) = D(\mathbf{x}) - A(\mathbf{x})$. We will now use a result that can be found in various sources such as [2] and [24].

Lemma 1.2 (Matrix-tree theorem)**.** *Kir*$(G, \boldsymbol{x})$ *of a graph is equal to the determinant of the Laplacian matrix $L(\boldsymbol{x})$ with one row and column deleted.*

For example, using the graph in Figure 1, we find that the Laplacian is:

$$\begin{pmatrix} x_1 + x_4 + x_5 & -x_5 & -x_1 & -x_4 \\ -x_5 & x_2 + x_5 & -x_2 & 0 \\ -x_1 & -x_2 & x_1 + x_2 + x_3 & -x_3 \\ -x_4 & 0 & -x_3 & x_3 + x_4 \end{pmatrix}$$

By deleting a row and column, for example the third, and calling this reduced matrix $L_3(\mathbf{x})$ we have that

$$\begin{aligned} \det(L_3(\mathbf{x})) &= ((x_1 + x_4 + x_5)(x_2 + x_5) - x_5^2)(x_3 + x_4) - (x_2 + x_5)x_4^2 \\ &= x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_5 + x_1x_4x_5 + x_2x_3x_4 + x_2x_3x_5 + x_2x_4x_5 + x_3x_4x_5 \\ &= \mathrm{Kir}(G, \mathbf{x}) \end{aligned}$$

We can now proceed with the proof of the above proposition.

Proof. Call $\mathbf{x} = \{x_e\}_{e \in E}$. $L(\mathbf{x})$ is the Laplacian matrix with each variable x_e assigned to edge e. Namely $L(\mathbf{x}) = BXB^T$, where B is the directed edge vertex matrix for any orientation of G and X is a diagonal matrix of the x_e values.

Delete the nth row and column of the Laplacian matrix. Call this new matrix $L_n(\mathbf{x})$. By the matrix-tree theorem, we have that if $\text{Kir}(G,\mathbf{x})$=0 for some $\mathbf{x}$ in $\mathbb{C}^n_{++}$, then $\det(L_n(\mathbf{x})) = 0$. Therefore there must be a nonzero vector ϕ such that $\phi L_n(\mathbf{x}) = 0$. We can extend this by adding a 0 in the nth column. Call this extended vector $(\phi, 0)$. We must then have that $(\phi, 0)L(G, \mathbf{x}) = (0, s)$ for some $s \in \mathbb{C}$. Therefore

$$(\phi, 0)L(\mathbf{x})(\phi, 0)^* = 0.$$

Moreover

$$(\phi, 0)L(\mathbf{x})(\phi, 0)^* = (\phi, 0)BXB^T(\phi, 0)^* = \sum_{e \in E} \left|\left(B^T(\phi, 0)^*\right)\right|^2 x_e.$$

Because G is connected, we know that B is invertible, so $B^T(\phi, 0)^* \neq 0$. Therefore the only way that this sum can be 0 is if there is some e such that $\text{Re}(x_e) \leq 0$, meaning that $\mathbf{x} \notin \mathbb{C}_{++}$ and $\text{Kir}(G, \mathbf{x})$ is real stable.

□

One final example will require use of Hurwitz's theorem [26].

Theorem 1.3 (Hurwitz's theorem)**.** *Call $U \subset \mathbb{C}^m$ a connected open set. Call $f_n : n \in \mathbb{N}$ a sequence of analytic functions that are non-vanishing on U. If the f_n converge to a limit function f on compact subsets of U, then f is non-vanishing on U or identically zero.*

An $n \times n$ matrix A is called *Hermitian* if $a_{i,j} = \overline{a_{j,1}}$. Namely, A is equal to its own conjugate transpose. Moreover, an $n \times n$ Hermitian matrix A is called *positive semidefinite* if $\mathbf{x}^T A\mathbf{x} \in \mathbb{R}_+$ for any $1 \times n$ column vector x with real components.

Proposition 1.4. *For $1 \leq i \leq m$, assign an $n \times n$ matrix A_i and a variable x_i. Call $\boldsymbol{x} = (x_1, \ldots, x_n)$. If each A_i is positive semidefinite and B is a Hermitian matrix, and if we define $f(\boldsymbol{z})$ as*

$$f(\boldsymbol{z}) = \det(A_1 z_1 + A_2 z_2 + \ldots + A_m z_m + B)$$

then $f(\boldsymbol{z})$ is stable in the upper half of the complex plane. Namely if $Im(z_i) > 0$ for $1 \leq i \leq m$, then $f(\boldsymbol{z}) \neq 0$.

Proof. Call $\bar{f}$ the coefficient-wise complex conjugate of f. We know that $\bar{A}_i = {A_i}^T$ and $\bar{B} = B^T$, so we know that $f = \bar{f}$. Therefore f is a real polynomial. Because we can write each A_i as the limit of positive definite matrices, we need only prove that if each of the A_i is positive definite then f is real stable, and then proceed by using Hurwitz' theorem. Consider vector $\mathbf{z} \in \mathbb{C}^m_{++} = \mathbf{a} + \mathbf{b}i$.

Define $Q = \sum^m{}_{i=1} b_i A_i$ and $H = \sum^m{}_{i=1} a_i A_i + B$. Because Q is positive definite it also has positive definite square-root. H is Hermitian, so

$$f(\mathbf{z}) = \det(Q)\det(iI + Q^{-1/2}HQ^{-1/2}).$$

Since $\det(Q) \neq 0$, if $f(\mathbf{z}) = 0$, then $-i$ is an eigenvalue of $Q^{-1/2}HQ^{-1/2}$. However this is impossible as $Q^{-1/2}HQ^{-1/2}$ cannot have imaginary eigenvalues. Therefore $f(\mathbf{z}) \neq 0$, so f is stable in the upper half of the complex plane.

□

1.2 Transformations

Stable polynomials are useful tools, as they maintain their stability under many transformations with each other. Here are two important but quick properties that we will use later on.

Theorem 1.5. *Call f and g stable polynomials and $\lambda \in \mathbb{C}^*$.*

(1) fg is stable

(2) λf is stable

Proof. For (1), note that the roots of fg are simply the roots of f and the roots of g. Therefore if f and g are stable, then so is fg.

For (2), λf has the same roots as f. $\square$

For composition of functions, if f and g are stable and we desire $f(g(x))$ to also be stable, we must guarantee that g preserves the right half plane. For example if g is the function $g(z) = 1/z$, then as $g(\mathbb{C}_{++}) = \mathbb{C}_{++}$, we know that $f(g(z))$ is real stable.

1.2.1 Real-part-positive functions

Call D a domain in $\mathbb{C}^n$ and $f : D \to \mathbb{C}$ a function analytic on D. f is called *real-part-positive* if $\mathrm{Re}(f(x)) \geq 0$ for all $x \in D$. Similarly, f is called *strictly real-part-positive* if $\mathrm{Re}(f(x)) > 0$.

Proposition 1.6. *Call D a domain in $\mathbb{C}^n$. Define a function $f : D \to \mathbb{C}$ which is analytic and real-part-positive. Then either f is strictly real-part-positive or f is an imaginary constant.*

Proof. By the open mapping theorem, we know that $f(D)$ is either open or f is constant. If $f(D)$ is open, then the image must be contained in $\mathbb{C}_{++}$ and therefore f is strictly real-part-positive. If $f(D)$ is a constant, then either $\mathrm{Re}(f(D)) = 0$, meaning f is purely imaginary, or $\mathrm{Re}(f(D) > 0)$, meaning f is strictly real-part-positive. $\square$

Proposition 1.7. *Call D a domain in $\mathbb{C}^n$. Define f and g as analytic on D. Assume that g is nonvanishing on D and f/g is real-part-positive on D. Then either f is non vanishing or identically zero. In particular, call P and Q polynomials in n variables with $Q \not\equiv 0$. If Q is stable and P/Q is real-part-positive on $\mathbb{C}^n_{++}$, then P is stable.*

Proof. By Proposition 1.6 we know that f/g is a constant function or is strictly real-part-positive. If it is a constant, then it is identically 0 (if $(f/g)(D) = 0$) or is nonvanishing if $(f/g)(D) = c$ where $c \neq 0$. If f/g is strictly real-part-positive, then f is nonvanishing. $\square$

However this property is not simply limited to when g is non vanishing. We call the set $Z(g)$ the set such that $z \in Z(g)$ if $g(z) = 0$. $Z(g)$ is a closed set and has empty interior if $g \not\equiv 0$, so we can use Proposition 1.7 on the open set $D \backslash Z(g)$. This may seem limiting, but by the following proposition, it is in fact general.

Proposition 1.8. *Call P and Q non trivial polynomials in n complex variables with P and Q relatively prime over $\mathbb{C}$. Call D a domain in $\mathbb{C}^n$. If P/Q is real part positive on $D \backslash Z(Q)$ then $Z(Q) \cap D = \emptyset$.*

Proof. Assume that there is a $z_0 \in D$ such that $Q(z_0) = 0$. If $P(z_0) \neq 0$, then Q/P is analytic in some neighborhood U of z_0 and is non constant, so by the open mapping theorem, $(Q/P)(U \backslash Z(Q))$ contains a neighborhood of 0. Call this neighborhood V. Therefore $(Q/P)(U \backslash Z(Q))$ contains $V \backslash \{0\}$, which contradicts the assumption that P/Q is real-part-positive on $D \backslash Z(Q)$.

If $P(z_0) = 0$, then from [23] 1.3.2 we know that for every neighborhood U of z_0, we have $(P/Q)(U \backslash Z(Q)) = \mathbb{C}$, which once again violates the assumption that P/Q is real-part-positive on $D \backslash Z(Q)$. □

2 The Permanent

Even though they are a simple concept, real stable polynomials prove useful in a variety of ways. What we will show here is not the first use of real stable polynomials, but nevertheless, this, the proof given by Leonid Gurvits concerning the van der Waerden's conjecture, provided a unique look at how to apply them.

The *permanent* of a square matrix A is as follows. Take a single element from each row of A. Multiply all of your chosen numbers together. Then add all possible permutations of this action. Formally, the permanent of a matrix A is

$$\text{per}A = \sum_{\pi \in S_n} \prod_{i=1}^{n} a_{i,\pi(i)}$$

for all possible S_n. For example, taking an $n \times n$ matrix where every element is 1, the permanent is $n!$, considering each product taken is 1 and there are $n!$ possible products.

The permanent, like the determinant, is a function that can be performed on square matrices. However, unlike the determinant, which can be discovered in polynomial time, finding the permanent is an NP-complete problem.

2.1 The van der Waerden Conjecture

The renowned 20th century mathematician Bartel Leendert van der Waerden, famous for writing the first modern algebra book, made a conjecture on the permanent in 1926. The van der Waerden Conjecture states that for a square matrix A that is non-negative and doubly stochastic, (namely the sum of each row and column is 1), the permanent is such that

$$\mathrm{per}(A) \geq \frac{n!}{n^n},$$

the minimum being reached when all elements are $1/n$. This remained an unsolved problem for 50 years until a proof was conceived by Falikman and Egorychev in 1980 and 1981 respectively. However, in 2008 many were surprised by a simpler proof by Leonid Gurvits, who at the time was a researcher at Los Alamos National Laboratory.

Gurvits used a perhaps counterintuitive approach to find the lower bound, looking at a polynomial related to the permanent, using properties of real stability, and then proving the desired result about the original polynomial.

For a given matrix A, consider the polynomial p_A such that

$$p_A(x_1, \ldots, x_n) := \prod_{i=1}^{n} \left(\sum_{j=1}^{n} a_{i,j} x_j \right).$$

Notice that the coefficient of $x_1 x_2 \cdots x_n$ is exactly $\mathrm{per}(A)$. In other words, if we define $\mathbf{x} = (x_1, \ldots, x_n)$,

$$\left. \frac{\partial^n p(\mathbf{x})}{\partial_{x_1} \partial_{x_2} \ldots \partial_{x_n}} \right|_{x_1 = \ldots = x_n = 0} = \mathrm{per} A.$$

Therefore, our main objective is to deduce the desired lower bound on the above derivative.

We now define a quantity called the *capacity*. The capacity of a polynomial p is defined such that

$$\mathrm{cap}(p) = \inf p(\mathbf{x})$$

taken over $\mathbf{x} \in \mathbb{R}^n$ such that $\prod_{i=1}^n x_i = 1$.

Lemma 2.1. *If A is doubly stochastic, then cap$(p_A) = 1$.*

Proof. We shall use the geometric-arithmetic mean inequality. Namely if $\lambda_1, \ldots, \lambda_n$, $x_1, \ldots, x_n \in \mathbb{R}_+$, and $\sum_{i=1}^n \lambda_i = 1$, then

$$\sum_{i=1}^{n} \lambda_i x_i \geq \prod_{i=1}^{n} x_i^{\lambda_i}.$$

We can now quickly deduce

$$p_A(x) = \prod_{i=1}^{n} \left(\sum_{j=1}^{n} a_{i,j} x_j \right) \geq \prod_i \prod_j x_j^{a_{i,j}} = \prod_j \prod_i x_j^{a_{i,j}} = \prod_j x_j^{\sum_i a_{i,j}} = \prod_j x_j = 1.$$

Therefore $\mathrm{cap}(p_A) \geq 1$. However $p_A(1, \ldots, 1) = 1$, so $\mathrm{cap}(p_A) = 1$.

□

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Figure 2: The polynomial $p(x)$ takes the product of the inner product of each row of A with $(x_1, \ldots x_n)^T$.

We define another quantity q_i for $0 \leq i \leq n$.

$$q_i(x_1, \ldots, x_i) = \left. \frac{\partial^{n-i} p_A}{\partial x_{i+1} \cdots \partial x_n} \right|_{x_{i+1} = \cdots = x_n = 0}.$$

Note that $q_0 =$per(A) and $q_n = p_A(\mathbf{x})$.

The method we shall use is to prove that

$$\text{per}(A) \geq \prod_{i=1}^{n} g(\min\{i, \lambda_A(i)\})$$

where $g(0) = 1$ and $g(k) = (\frac{k-1}{k})^{k-1}$ for $k = 1, 2, \ldots$ and $\lambda_A(i)$ is the number of non zeros in the ith column of A.

For the polynomial $p(x_1, \ldots, x_k)$, define the quantity

$$p'(x_1, \ldots, x_{k-1}) = \left. \frac{\partial p}{\partial x_k} \right|_{x_k = 0}.$$

Note that $q_{i-1} = q_i'$.

Before we can prove Gurvits' inequality, we must prove an important relationship between cap(p) and cap(p').

Theorem 2.2. *Call $p \in \mathbb{R}_+[x_1, \ldots, x_n]$ a real stable polynomial that is homogenous of degree n. Then either $p' \equiv 0$ or p' is real stable. Moreover cap$(p') \geq$ cap$(p)g(k)$, where k is the degree of x_n in p.*

In order to obtain this result, we must use the following lemma.

Lemma 2.3. *Define $p \in \mathbb{C}[x_1 \ldots, x_n]$ as a real stable and homogenous polynomial of degree m. Then if $x \in \mathbb{C}_+^n$, then $|p(x)| \geq |p(Re(x))|$.*

Proof. By continuity we can assume that $x \in \mathbb{C}_{++}^n$. Because p is real stable, we know that $p(\text{Re}(x)) \neq 0$. Therefore if we consider $p(x + s\text{Re}(x))$ a degree m polynomial in s, we can write it as

$$p(x + s\text{Re}(x)) = p(\text{Re}(x)) \prod_{i=1}^{m} (s - c_i)$$

for some $c_1, \ldots, c_m \in \mathbb{C}$. For each $1 \leq j \leq m$, we know that as $p(x + c_j\text{Re}(x)) = 0$, $x + c_j\text{Re}(x) \notin \mathbb{C}_{++}^n$ since p is real stable.

Call $c_j = a_j + b_j i$ for $a_j, b_j \in \mathbb{R}$. Then because $x + c_j\mathrm{Re}(x) \notin \mathbb{C}^n_{++}$, by looking at the real part of this expression we know that $(1 + a_j)\mathrm{Re}(x) < 0$, so $a_j < -1$. Therefore $|c_j| \geq 1$, so

$$|p(x)| = |p(x + 0\mathrm{Re}(x))| = |p(\mathrm{Re}(x))| \prod_{i=1}^{m} |c_i| \geq |p(\mathrm{Re}(x))|.$$

□

What we wish to prove is the following:

Theorem 2.4. *Let $p \in \mathbb{R}_+[x_1, \ldots, x_n]$ be a real stable polynomial that is homogenous of degree n. Then for y such that $\prod Re(y_i) = 1, y \in \mathbb{C}^{n-1}_{++}$*

(1) if $p'(y) = 0$ then $p' \equiv 0$.

(2) If y is real, then $cap(p)g(k) \leq p'(y)$

With this, p' is real stable or equivalent to 0, as by (2) it must be greater than or equal to 0 when y is real, and if $p'(y) = 0$ then by (1) $p' \equiv 0$. Therefore this theorem proves Theorem 2.2.

Proof. First we will prove that for $t \in \mathbb{R}_{++}$

$$\mathrm{cap}(p) \leq \frac{p(\mathrm{Re}(y), t)}{t}.$$

Call $\lambda = t^{-1/n}$ and $x = \lambda(\mathrm{Re}(y), t)$. Note that $\prod_{i=1}^n x_i = \lambda^n \left(\prod_{i=1}^{n-1} \mathrm{Re}(y_i)\right) t = 1$. Then, as p is homogenous of degree n, we have

$$\mathrm{cap}(p) \leq p(x) = \lambda^n p(\mathrm{Re}(y), t) = \frac{p(\mathrm{Re}(y), t)}{t}.$$

We will now prove the theorem in 3 different cases. For the first case, assume that $p(y, 0) = 0$. Then $0 = |p(y, 0)| \geq |p(\mathrm{Re}(y), 0)| \geq 0$. Therefore $p(\mathrm{Re}(y), 0) = 0$.

From this we have that

$$p'(y) = \lim_{t\to 0^+} \frac{p(y, t) - p(y, 0)}{t} = \lim_{t\to 0^+} \frac{p(y, t)}{t}$$

and

$$p'(\mathrm{Re}(y)) = \lim_{t\to 0^+} \frac{p(\mathrm{Re}(y), t) - p(\mathrm{Re}(y), 0)}{t} = \lim_{t\to 0^+} \frac{p(\mathrm{Re}(y), t)}{t}.$$

By Lemma 2.3, we know that $p(\mathrm{Re}(y), t) \leq |p(y, t)|$ for $t \geq 0$. Therefore

$$\mathrm{cap}(p) \leq \lim_{t\to 0^+} \frac{p(\mathrm{Re}(y))}{t} = p'(\mathrm{Re}(y)) \leq \lim_{t\to 0^+} \frac{|p(y, t)|}{t} = |p'(y)|.$$

Therefore, if $p'(y) = 0$, then $p'(\mathrm{Re}(y)) = 0$. As all of the coefficients of p are non-negative, this means that $p' \equiv 0$. Thus we have (1). We have (2) as $g(k) \leq 1$ regardless of k.

For the second case, assume that the degree of t in p is at most 1. This means that the degree of t in $p(\mathrm{Re}(y), t)$ is at most 1 as $p(\mathrm{Re}(y), t) \leq |p(y, t)|$. By L'Hôpital's Rule we know that

$$\lim_{t\to\infty} \frac{p(y,t)}{t} = p'(y) \text{ and } \lim_{t\to\infty} \frac{p(\mathrm{Re}(y),t)}{t} = p'(\mathrm{Re}(y)).$$

Therefore, using once again the result from above,

$$\mathrm{cap}(p) \leq \lim_{t\to\infty} \frac{p(\mathrm{Re}(y),t)}{t} = p'(\mathrm{Re}(y)) \leq \lim_{t\to\infty} \frac{|p(y,t)|}{t} = |p'(y)|.$$

(2) follows immediately. (1) does as well, as if $p'(y) = 0$ then $p'(\mathrm{Re}(y)) = 0$, so all the coefficients must be 0.

Finally the third case comprises all other cases, namely when $p(y, 0) \neq 0$ and $\deg_t(p) \geq 2$.

Because $p(y, 0)$ is nonzero, we can write $p(y, t)$ as a polynomial in t. By defining $k = \deg_t(p)$, we can rewrite the polynomial as

$$p(y,t) = p(y,0)\prod_{i=1}^{k}(1 + a_i t).$$

for some $a_1, \ldots, a_k \in \mathbb{C}$. Therefore $p'(y) = p(y, 0)\sum_{i=1}^{k} a_i$. Also because the degree of t is at least 2, we know that there must be at least one $a_j \neq 0$. Define the *cone* of a set of numbers such that

$$\mathrm{cone}\{y_1, \ldots, y_n\} = z \in \mathbb{C} : z = \sum_{i=0}^{n} c_i y_i \text{ for } c_i \in \mathbb{R}_+.$$

We now claim that $a_j^{-1} \in \mathrm{cone}\{y_1, \ldots, y_n\}$. Assume not. Then there is some $\lambda \in C$ such that $\mathrm{Re}(\lambda a_j^{-1}) < 0$ and $\mathrm{Re}(\lambda y_i) > 0$ for each $1 \leq i \leq n - 1$. Therefore $(\lambda y, -\lambda a_j^{-1}) \in \mathbb{C}^n_{++}$. However we know that

$$p(\lambda y, -\lambda a_j^{-1}) = \lambda^n p(y,0)\prod_{i=1}^{k}(1 - a_i a_j^{-1}) = 0$$

which is a contradiction as p is stable. Therefore $a_j^{-1} \in \mathrm{cone}\{y_1, \ldots, y_n\}$ and $\mathrm{Re}(a_j) > 0$. Thus $p'(y) = p(y, 0)\sum_{i=1}^{k} a_i \neq 0$ so in this case p' is real stable and gives us (1).

For (2), assume that y is real. Therefore each a_i is real non-negative. Let us assume that $p(y, 0) = 1$ and set $t = \frac{k}{(k-1)p'(y)}$.

We shall now use the geometric-arithmetic mean inequality one more time.

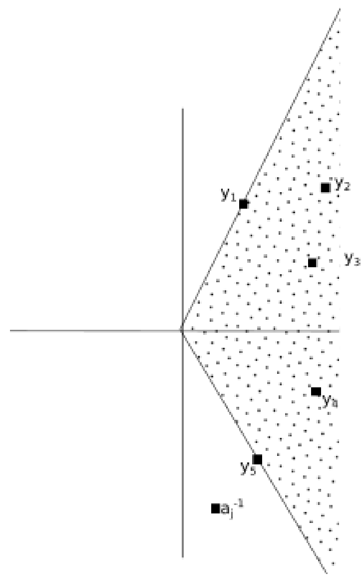

Figure 3: The non-negative linear combination of the y_i forms a cone, marked as the dotted region in the complex plane. Clearly if a_j^{-1} is not in the cone of $y_1 \dots y_{n-1}$ then we can find a $\lambda \in \mathbb{C}$ that would give us our desired rotation.

$$\begin{aligned} p(y,t) &= \prod_{i=1}^{k}(1+a_it) = \left(\sqrt[k]{\prod_{i=1}^{k}(1+a_it)}\right)^k \\ &\leq \left(\frac{1}{k}\sum_{i=1}^{k}(1+a_it)\right)^k = \left(\frac{1}{k}(k+p'(y)t)\right)^k = \left(1+\frac{1}{k-1}\right)^k = \left(\frac{k}{k-1}\right)^k. \end{aligned}$$

Therefore we have that

$$\mathrm{cap}(p) \leq \frac{p(y,t)}{t} \leq \frac{1}{t}\left(\frac{k}{k-1}\right)^k = p'(y)\left(\frac{k}{k-1}\right)^{k-1}$$

giving us the second desired property. □

Having proved this theorem, we can now apply this directly to the permanent.

Theorem 2.5. *Call λ_A the number of nonzero entries in the ith column of A. If A is a non-negative doubly stochastic matrix, then*

$$per(A) \geq \prod_{i=1}^{n} g(\min\{i, \lambda_A(i)\}).$$

Moreover,

$$per(A) \geq \frac{n!}{n^n}.$$

Proof. Firstly note that $p_A(x) = 0$ implies that $\langle a_i, x\rangle = 0$ for some i, meaning that there is a zero row. Because A is doubly stochastic we know this cannot be the case, therefore p_A is real stable.

In q_i, all nonzero terms that contain x_i must be of the form $x_1^{\alpha_1} \cdots x_i^{\alpha_i} x_{i+1} \cdots x_n$ in the polynomial $p_A(x)$. As $p_A(x)$ is homogenous of degree n, $\alpha_i = \deg_{x_i}(q_i) \leq i$. Similarly in p_A we know that $\deg_{x_i}(q_i) \leq \lambda_A$, as a higher degree would involve the multiplication of a 0 term and therefore have coefficient 0. Moreover, g is monotone nonincreasing.

Therefore for $1 \leq i \leq n$,

$$\operatorname{cap}(q_{i-1}) \geq \operatorname{cap}(q_i) g(\deg_{x_i}(q_i)) \geq \operatorname{cap}(q_i) g(\min\{i, \lambda_A(i)\}).$$

Once again $p_A = q_n$ and $\operatorname{per}A = q_0$. Moreover, from Theorem 2.1, $\operatorname{cap}(p_A) = 1$. Therefore by taking the above inequality iteratively n times, we have

$$\operatorname{per}(A) \geq \prod_{i=1}^{n} \left(\frac{i-1}{i}\right)^{i-1} = \prod_{i=1}^{n} i \frac{(i-1)^{i-1}}{i^i} = \frac{n!}{n^n}$$

giving us the desired lower bound on the permanent.

□

3 Ramanujan Graphs

3.1 Graph Overview

We now move from the concept of matrices and the permanent into another area of combinatorics, namely that of graph theory. Srinivasa Ramanujan, one of the most accomplished mathematicians of the early 20th century, discovered a number of results without the benefit of a formal mathematical education. The result that we will focus on is not in fact discovered by Ramanujan, but was based on his work. This is the concept of the *Ramanujan graph.*

Call a graph $G = (V, E)$ *d–regular* if the number of edges incident to each vertex $v \in V$ is d. Moreover a graph is *bipartite* if there exist subsets $X, Y \subset V$ such that $X \cup Y = V$ and $E \subseteq X \times Y$. Namely each edge in E is between a vertex in X and a vertex in Y.

For a d–regular graph G, its adjacency matrix $A(G)$ always has the eigenvalue d. If G is bipartite, then $-d$ is also an eigenvalue of A. We call d (and $-d$ if G is bipartite) the *trivial eigenvalues* of G. G is then called a *Ramanujan graph* if all of its non trivial eigenvalues are between $-2\sqrt{d-1}$ and $2\sqrt{d-1}$.

Ramanujan graphs are examples of *expander graphs*, which, intuitively speaking, are graphs for which every small subset of vertices has a large set of vertices

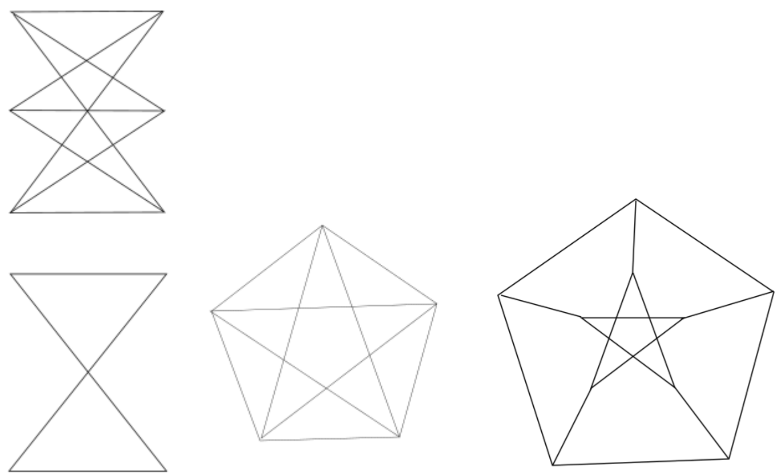

Figure 4: Four examples of Ramanujan graphs

adjacent to it. Practical applications of this therefore include telephone and telegraph wiring. Interestingly, one of the ways that these graphs were first used was to examine the ability for codes to be transmitted over noisy channels [17].

One question proposed is whether there is an infinite family of Ramanujan graphs with a set degree d. It turns out that in fact there are infinite families of bipartite Ramanujan graphs, and that the Ramanujan bound is the tightest on the eigenvalues of the adjacency matrix that we can make for an infinite regular family. One method of finding an infinite family of bipartite graphs uses real stable polynomials. The advantage to finding an infinite family of bipartite graphs is that the eigenvalues of the adjacency matrix of a bipartite graph are symmetric around 0. Therefore we only need to prove one side of the bound. For this chapter, we shall consider stability over the upper half plane as opposed to the right half plane. Namely, for a polynomial $p(z_1, \ldots, z_n)$, p is referred to as stable if $p(z_1, \ldots, z_n)$ when $\mathrm{Im}(z_i) > 0$ for $1 \leq i \leq n$.

3.2 Bound of an Infinite Family

As motivation, we will show that this is in fact the tightest such that we can create an infinite family of graphs.

Let $G = (V, E)$ be a regular graph. $\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_{n-1}$ represent the eigenvalues of the adjacency matrix of G. We use Theorem 1 from [20].

Theorem 3.1. *Let $G = (V, E)$ be a graph of maximum degree d such that the distance between two sets of edges is at least $2k + 2$. Then*

$$\lambda_1 \geq 2\sqrt{d-1}\left(1 - \frac{1}{k+1}\right) + \frac{1}{k+1}.$$

Given this theorem, we can show that Ramanujan graphs provide the smallest infinite family of d–regular graphs.

Corollary 3.2. *Call $\mathcal{G}_n^d$ an infinite family of d–regular graphs with n vertices. Then*

$$\lim_{n\to\infty} \inf_{G\in\mathcal{G}_n^d} \lambda_1 \geq 2\sqrt{d-1}.$$

Moreover Ramanujan graphs provide the smallest infinite family of d–regular graphs.

Proof. As $n \to \infty$ but d stays the same, the highest distance between two series of edges goes towards infinity. With $n \geq 2d^{2k+1}+1$, take an edge $e_1 \in V$. As G is d–regular, there are at most $2d^{2k+1}$ vertices that are distance $2k+1$ away from v_1. Thus there exists a vertex v_1' that is at least distance $2k+2$ from e_1. Call an edge of v_1' e_1'. Now take an edge that has vertices apart from those of e_1 and e_1'. This must also have a corresponding edge of at least distance $2k+2$. Therefore as $n \to \infty$ we can take $k \to \infty$, meaning that

$$\lim_{n\to\infty} \inf_{G\in\mathcal{G}_n^d} \lambda_1 \geq 2\sqrt{d-1}.$$

Therefore, the Ramanujan bound of $|\lambda| \leq 2\sqrt{d-1}$ for λ a nontrivial eigenvalue is in fact the best possible bound. □

3.3 Bilu Linial Covers

We will now use as background an important result from Bilu and Linial concerning the eigenvalues of a covering of a graph, and then apply this to Ramanujan graphs.

Consider two graphs $\hat{G}$ and G. A map $f : \hat{G} \to G$ is called a *covering map* if f is surjective and *locally isomorphic*, namely for each $v \in \hat{G}$ there is a neighborhood $U \ni v$ such that $f(U)$ is an isomorphism. We call $\hat{G}$ a *covering graph* if there exists such a map $f(z)$. We can also call $\hat{G}$ a *lift* of G. More specifically, we can call $\hat{G}$ an $n-lift$ of G if $\forall v \in G$, the *fiber* of v $f^{-1}(v)$ has n elements.

We wish to find a covering of G that has eigenvalues satisfying the Ramanujan property. To do so we will introduce a way to create a 2–lift of a graph. A *signing* $s : E \to \{1, -1\}$ of G is an assignation of either a positive or negative value to each edge. We define the adjacency matrix $A_s(G)$ such that the entries of A_s are $s(\{i,j\})$ if $\{i,j\}$ is an edge in A. Otherwise the entries are 0.

From Bilu and Linial, we find a 2–lift of a graph G in the following way. Consider two copies of the original graph G_1 and G_2 and a signing of the graph G. The edges in the fiber of edge $\{x,y\}$ are $\{x_0,y_0\}$ and $\{x_1,y_1\}$ if $s(\{x,y\}) = 1$, but are $\{x_0,y_1\}$ and $\{x_1,y_0\}$ if $s(\{x,y\}) = -1$.

We now use the important result from Bilu and Linial [1]:

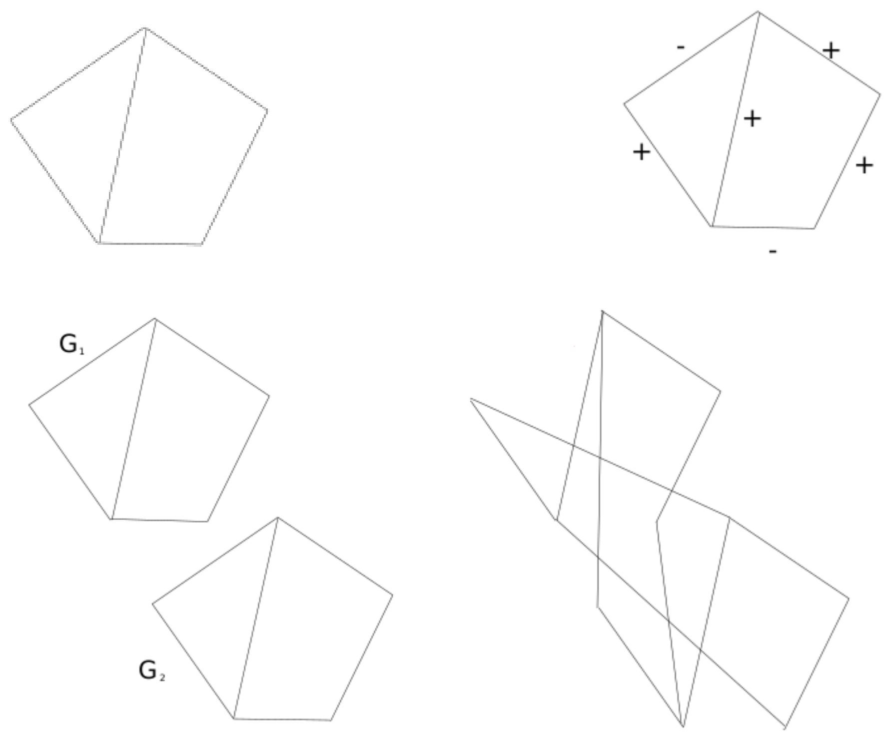

Figure 5: An example of the Bilu Linial cover. We take two copies of the original graph. All the edges assigned 1 in the original graph are preserved. If the edge is assigned -1, then we delete the edge, and then connect the corresponding vertices from one graph to the other.

Lemma 3.3 (Bilu and Linial)**.** *Call G a graph with adjacency matrix $A(G)$. Call s a signing with adjacency matrix $A_s(G)$, associated with a 2–lift $\hat{G}$. Every eigenvalue of A and every eigenvalue of A_s are eigenvalues of $\hat{G}$, and the multiplicity of the eigenvalues of $\hat{G}$ is the sum of the multiplicities in A and A_s.*

Proof. Consider
$$2A(\hat{G}) = \begin{pmatrix} A + A_s & A - A_s \\ A - A_s & A + A_s \end{pmatrix}$$

Call u an eigenvector of A. Then (u, u) is an eigenvector of $\hat{G}$ with the same eigenvalue. Call u_s an eigenvector of $A_s(G)$. Then $(u_s, -u_s)$ is an eigenvector of $\hat{G}$ with the same eigenvalue. $\square$

This will be the first tool we use towards our final result.

3.4 Interlacing

To prove that there is an infinite family by using real stable polynomials, we must first provide a series of lemmas and definitions.

Firstly, we define the matching polynomial as follows. A *matching M* is a subset $E' \subset E$ such that there do not exists two edges $e_1, e_2 \in E'$ such that $e_1 = \{v_1, v_1'\}, e_2 = \{v_2, v_2'\}$ such that v_1 or $v_1' = v_2$ or v_2'. Namely, a matching is a set of edges that do not contain common vertices. Call m_i the number of matchings of a graph G with i edges.

We define the *matching polynomial* as
$$\mu_G(x) = \sum_{i \geq 0} (-1)^i m_i x^{n-2i}.$$

We also must introduce the concept of *interlacing*, a relation between two polynomials.

Call $f(z)$ and $g(z)$ two real rooted polynomials of degree d. We say $g(z)$ *interlaces* $f(z)$ if the roots of the two polynomials alternate, with the lowest root of g lesser than the lowest root of f. Namely if the roots of f are $r_1 \leq \ldots \leq r_d$ and the roots of g are $s_1 \leq \ldots \leq s_d$, g interlaces f if
$$s_1 \leq r_1 \leq s_2 \leq r_2 \leq \ldots \leq s_d \leq r_d.$$

Also, if there is a polynomial that interlaces two functions f and g, then we say they have a *common interlacing*. Although the following result from 3.51 of Fisk's book *Polynomials, Roots and Interlacing* [10] is relatively straightforward, it features much computation:

Lemma 3.4. *Call $f(z)$ and $g(z)$ two real polynomials of the same degree such that every convex combination is real-rooted. Then $f(z)$ and $g(z)$ have a common interlacing.*

The second lemma we will use towards our objective is the following:

Lemma 3.5. *Call $f(z)$ and $g(z)$ two real polynomials of the same degree that have a common interlacing and positive largest coefficients. The largest root of $f(z)+g(z)$ is greater than or equal to one of the largest roots of $f(z)$ and $g(z)$.*

Proof. Call $h(z)$ the common interlacing of $f(z)$ and $g(z)$. If α is the largest root of f, β is the largest root of g and γ is the largest root of h, then we know that $\gamma \leq \alpha$ and $\gamma \leq \beta$. Because f and g have positive largest coefficient, as $z \to \infty$ then $f, g \to \infty$. Therefore $f+g > 0$ for all $z \geq \max\{\alpha, \beta\}$. The second largest roots of f and g are both at most γ, so here $f(\gamma) \leq 0$ and $g(\gamma) \leq 0$. Therefore $f(z)+g(z) \leq 0$ for $z \in [\gamma, \min\{\alpha, \beta\}]$, and either α or β is at most the largest root of $f+g$. $\square$

For $k < m$, a set $S_1 \times \ldots, \times, S_m$, and assigned values $\sigma_1 \in S_1, \ldots, \sigma_m \in S_m$, we define the polynomial

$$f_{\sigma_1,\sigma_2,\ldots,\sigma_k} = \sum_{\sigma_{k+1},\ldots,\sigma_m \in S_{k+1}\times\cdots\times S_m} f_{\sigma_1,\ldots,\sigma_k,\sigma_{k+1},\ldots,\sigma_m}.$$

We also define

$$f_0 = \sum_{\sigma_1,\ldots,\sigma_m \in S_1\times\cdots\times S_m} f_{\sigma_1,\ldots,\sigma_m}.$$

We call the polynomials $\{f_{\sigma_1,\ldots,\sigma_m}\}_{S_1,\ldots,S_m}$ an *interlacing family* if for all k such that $0 \leq k \leq m-1$ and for all $(\sigma_1, \ldots, \sigma_k) \in S_1 \times \cdots \times S_k$, the polynomials $\{f_{\sigma_1,\ldots,\sigma_k,\tau}\}_{\tau \in S_{k+1}}$ have a common interlacing.

Theorem 3.6. *Call $\{f_{\sigma_1,\ldots,\sigma_m}\}$ an interlacing family of polynomials with positive leading coefficient. Then there is some $\sigma_1, \ldots, \sigma_m \in S_1, \ldots, S_m$ such that the largest root of $f_{\sigma_1,\ldots,\sigma_m}$ is less than the largest root of f_0.*

Proof. The f_{α_1} for $\alpha_1 \in S_1$ are interlacing, so by 3.5, we know that one of the polynomials has a root at most the largest root of f_0. Proceeding inductively we see that for every k and any $\sigma_1, \ldots, \sigma_k$ there is a choice of α_{k+1} such that the largest root of $f_{\sigma_1,\ldots,\sigma_k,\alpha_{k+1}}$ is at most the largest root of $f_{\sigma_1,\ldots,\sigma_k}$. $\square$

3.5 The Upper Half Plane

Consider the family of polynomials $f_\sigma = \det(xI - A_\sigma)$ taken over all possible signings σ of the adjacency matrix $A(G)$. We now wish to show these polynomials form an interlacing family. The easiest way to do this is to prove that common interlacings are equivalent to qualities of real-rooted polynomials. Thus we will show that for all $p_i \in [0,1]$, the following polynomial is real rooted:

$$\sum_{\sigma \in \{\pm 1\}^m} \left(\prod_{i:\sigma_i=1} p_i \right) \left(\prod_{i:\sigma_i=-1} (1-p_i) \right) f_\sigma(x).$$

This result would be equivalent towards finding an interlacing family if we utilize Theorem 3.6 from [7]:

Lemma 3.7. *Let $f_1, f_2, \dots, f_k$ be real rooted polynomials with positive leading coefficients. These polynomials have a common interlacing if and only if*

$$\sum_{i=1}^{k} \lambda_i f_i$$

is real rooted for all $\lambda_i \geq 0, \sum_{i=1}^{k} \lambda_i = 1$.

We will begin in the following manner

Lemma 3.8. *Given an invertible matrix A, two vectors a and b, and $p \in [0,1]$, it is the case that*

$$(1 + p\partial_u + (1-p)\partial_v) \det(A + uaa^T + vbb^T)\big|_{u=v=0} = p\det(A+aa^T)+(1-p)\det(A+bb^T).$$

Proof. The matrix determinant lemma, from various sources such as [15] tells us that for a non-singular matrix and real value t,

$$\det(A + taa^T) = \det(A)(1 + ta^T A^{-1} a)$$

By taking a derivative in respect to t, we obtain Jacobi's formula saying that

$$\partial_t \det(A + taa^T) = \det(A)(a^T A^{-1} a).$$

Therefore

$$(1 + p\partial_u + (1-p)\partial_v) \det(A + uaa^T + vbb^T)\big|_{u=v=0} \tag{1}$$
$$= \det(A)(1 + p(a^T A^{-1} a) + (1-p)(b^T A^{-1} b)) \tag{2}$$

By the matrix determinant lemma, this quantity equals

$$p\det(A + aa^T) + (1-p)\det(A + bb^t)$$

□

We then use one of the main results from [3].

Lemma 3.9. *Call $T : \mathbb{R}[z_1, \dots, z_n] \to \mathbb{R}[z_1, \dots, z_n]$ such that*

$$T = \sum_{\alpha,\beta \in \mathbb{N}^n} c_{\alpha,\beta} z^\alpha \partial^\beta$$

where $c_{\alpha,\beta} \in \mathbb{R}$ and is zero for all but finitely many terms. Then call

$$F_T(z,w) = \sum_{\alpha,\beta} z^\alpha w^\beta.$$

T preserves real stability if and only if $F_T(z, -w)$ is real stable.

From this we can find a useful corollary.

Corollary 3.10. *For $r, s \in \mathbb{R}_+$ and variables u and v, the polynomial $T = 1 + r\partial_u + s\partial_v$ preserves real stability.*

Proof. We need only show that $1 - ru - sv$ is real stable. To see this, consider if u and v have positive imaginary part. Then $1 - ru - sv$ has negative imaginary part, so it is necessarily non-zero. $\square$

Theorem 3.11. *Call $a_1, \ldots, a_m$ and $b_1, \ldots, b_m$ vectors in $\mathbb{R}^n$, $p_1, \ldots p_m$ real numbers in $[0, 1]$, and D a positive semidefinite matrix. Then*

$$P(x) = \sum_{S \subseteq [m]} \left(\prod_{i \in S} p_i \right) \left(\prod_{i \notin S} (1 - p_i) \right) \det \left(xI + D + \sum_{i \in S} a_i a_i{}^T + \sum_{i \notin S} b_i b_i{}^T \right)$$

is real rooted.

Proof. We define

$$Q(x, u_1, \ldots, u_m, v_1, \ldots, v_m) = \det \left(xI + D + \sum_i u_i a_i a_i{}^T + \sum_i v_i b_i b_i{}^T \right).$$

By Proposition 1.4, Q is real stable.

We want to show that if $T_i = 1 + p_i \partial_{u_i} + (1 - p_i)\partial_{v_i}$ then

$$P(x) = \left(\prod_{i=1}^{m} T_i|_{u_i = v_i = 0} \right) Q(x, u_1, \ldots, u_m, v_1, \ldots, v_m)$$

In order to do this, we will show that

$$\left(\prod_{i=1}^{k} T_i|_{u_i = v_i = 0} \right) Q(x, u_1, \ldots, u_m, v_1, \ldots, v_m)$$

equals

$$\sum_{S \subseteq [k]} \left(\prod_{i \in S} p_i \right) \left(\prod_{i \in [k] \setminus S} (1 - p_i) \right) \det \left(xI + D + \sum_{i \in S} a_i a_i{}^T + \sum_{i \in [k] \setminus S} b_i b_i{}^T + \sum_{i > k} (u_i a_i a_i{}^T + v_i b_i b_i{}^T) \right)$$

We will do so by induction on k. When $k = 0$ this is the definition of Q. The inductive step is proved using Lemma 3.8. When we induct up to the case where $k = m$ we have proved the desired equality.

If we consider the stable function $f(x_1, \ldots, x_n)$, clearly the function $f(x_1, \ldots x_{n-1})|_{x_n = z}$ is real stable if $\mathrm{Im}(z) > 0$. Therefore if we take $z \to 0$, Hurwitz's theorem implies that if we set some variables of f to 0, we will maintain real stability or the function will be zero everywhere. We then can use Corollary 3.10 to show that $P(x)$ is real stable. As $P(x)$ is real stable and a function of one variable, it is real rooted.

$\square$

Showing that $P(x)$ is real stable gives us the following result:

Theorem 3.12. *The following polynomial is real rooted:*

$$\sum_{\sigma\in\{\pm1\}^m}\left(\prod_{i:\sigma_i=1}p_i\right)\left(\prod_{i:\sigma_i=-1}(1-p_i)\right)f_\sigma.$$

Proof. Call d the maximum degree of G. We will prove that

$$R(x)=\sum_{\sigma\in\{\pm1\}^m}\left(\prod_{i:\sigma_i=1}p_i\right)\left(\prod_{i:\sigma_i=-1}(1-p_i)\right)\det(xI+dI-A_s)$$

is real rooted, which would imply that our original polynomial is as well, considering their roots differ by d. Notice that $dI-A_s$ is a signed Laplacian matrix plus a non-negative diagonal matrix. Call e_u the vector with a 1 in the ith row, where i is the row associated with the vertex u in A_s. We then define the $n\times n$ matrices $L^1_{u,v}=(e_u-e_v)(e_u-e_v)^T$ and $L^{-1}_{u,v}=(e_u+e_v)(e_u+e_v)^T$. Call $\sigma_{u,v}$ the sign attributed to the edge $\{u,v\}$. We then have

$$dI-A_s=\sum_{\{u,v\}\in E}L^{\sigma_{u,v}}_{u,v}+D$$

where D is the diagonal matrix where the ith diagonal entry is $d-d_u$, where u is the ith column in A_s. D is non-negative, so it is positive semidefinite. We set $a_{u,v}=(e_u-e_v)$ and $b_{u,v}=(e_u+e_v)$. Therefore $R(x)$ is

$$\sum_{\sigma\in\{\pm1\}^m}\left(\prod_{i:\sigma_i=1}p_i\right)\left(\prod_{i:\sigma_i=-1}(1-p_i)\right)\det\left(xI+D+\sum_{\sigma_{u,v}=1}a_{u,v}a_{u,v}^T+\sum_{\sigma_{u,v}=1}b_{u,v}b_{u,v}^T\right)$$

Therefore, $R(x)$ must be real rooted by Theorem 3.11, so our original function is also real rooted.

$\square$

Corollary 3.13. *The polynomials $\{f_\sigma\}_{\sigma\in\{\pm1\}^m}$ form an interlacing family.*

Proof. We wish to show that for every assignment $\sigma_1\in\pm1,\ldots,\sigma_k\in\pm1$, $\lambda\in[0,1]$

$$(\lambda f_{\sigma_1,\ldots,\sigma_k,1}+(1-\lambda)f_{\sigma_1,\ldots,\sigma_k,-1})\,(x)$$

is real-rooted.

However to show this we merely use Theorem 3.12 with $p_1=(1+\sigma_i)/2$ for $1\le i\le k$, $p_{k+1}=\lambda$ and $p_{k+2},\ldots,p_m=1/2$. $\square$

We can now provide the finishing touches.

Lemma 3.14. *Call $K_{c,d}$ a complete bipartite graph. Every non-trivial eigenvalue of $K_{c,d}$ is 0.*

Proof. The adjacency matrix of this graph has trace 0 and rank 2, so besides the necessary eigenvalues of $\pm\sqrt{cd}$ all eigenvalues must be 0. $\square$

For an n vertex graph G we now define the *spectral radius* $\rho(G)$ such that

$$\rho(T) = \max\{|\lambda_1|, \ldots, |\lambda_n|\}$$

where the λ_i are the eigenvalues of $A(G)$.

We borrow our preliminary result from [16] theorems 4.2 and 4.3, who showed the following.

Lemma 3.15 (Heilmann)**.** *Call G a graph with universal cover T. Then the roots of μ_G are real and have absolute value at most $\rho(T)$.*

For the final pieces, we will use two results from C. Godsil. First, we now use a result from Godsil's book *Algebraic Combinatorics*, namely Theorem 5.6.3 from [11].

Proposition 3.16. *Let T be a tree with maximum degree d. Then $\rho(T) < 2\sqrt{d-1}$.*

For the second result, we can consider the characteristic polynomial of a signing of a graph G. By averaging over all potential signings, we obtain a value known as the *expected characteristic polynomial.* The second important result from Godsil is Corollary 2.2 from [13].

Proposition 3.17. *The expected characteristic polynomial of $A_s(G)$ is $\mu_G(x)$.*

Theorem 3.18. *Call G a graph with adjacency matrix A and universal cover T. There is a signing s of A such that all of the eigenvalues of the corresponding matrix A_s are at most $\rho(T)$. Moreover if G is d–regular, there is a signing s such that the eigenvalues are at most $2\sqrt{d-1}$.*

Proof. By Corollary 3.13, there must be a signing s with roots at most those of μ_G, and by Lemma 3.15 we know that these roots are at most $\rho(T)$. The second part follows as the covering of a d–regular graph is the infinite d–regular tree, which has spectral radius of $2\sqrt{d-1}$ from Proposition 3.16. $\square$

Theorem 3.19. *For $d \geq 3$, there is an infinite sequence of d–regular bipartite Ramanujan graphs.*

Proof. By Lemma 3.14 we know that $K_{d,d}$ is Ramanujan. Call G a d–regular bipartite Ramanujan graph. By Theorem 3.18 and Lemma 3.3 we know that every d–regular bipartite graph has a 2–lift such that every non-trivial eigenvalue has absolute eigenvalue at most $2\sqrt{d-1}$. The 2–lift of a bipartite graph must be bipartite and the eigenvalues of a bipartite graph must be centered around 0, therefore this 2–lift is a regular bipartite Ramanujan graph.

Therefore by starting at $K_{d.d}$, we can create an infinite family of Ramanujan graphs by taking the 2–lift of the previous graph to create a new one with twice as many vertices that still satisfies the Ramanujan property and is d–regular bipartite. $\square$

We have thus found our infinite family of Ramanujan graphs. Once again, although the method is slightly more haphazard, we see that the unique properties of real stable polynomials have lead us to a surprising result of combinatorics.

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