Crepant resolutions of $\mathbb{Q}$-factorial threefolds with compound Du Val singularities

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Abstract

We study the set of crepant resolutions of $\mathbb{Q}$-factorial threefolds with compound Du Val singularities. We derive sufficient conditions for the Kawamata–Kollár–Mori–Reid decomposition of the relative movable cone into relative ample cones to be the decomposition of a cone into chambers for a hyperplane arrangement. Under our sufficient conditions, the hyperplane arrangement can be determined by computing intersection products between exceptional curves and divisors on any single crepant resolution. We illustrate our results by considering the Weierstrass models of elliptic fibrations arising from Miranda collisions with non-Kodaira fibers. Many of our results extend to the set of crepant partial resolutions with $\mathbb{Q}$-factorial terminal singularities.

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for my parents:

Dr. Lalita Jagadeesan
Dr. Radhakrishan Jagadeesan

Contents

0 Preface

This thesis studies the geometry of mathematical objects called *singular algebraic varieties*. Algebraic varieties are the spaces of solutions to systems of polynomial equations.[1] We call one-dimensional and two-dimensional varieties *curves* and *surfaces*, respectively. The class of algebraic varieties include ubiquitous objects like lines, conic sections, and elliptic curves—as well as higher-dimensional objects like double cones, which we call *quadric cones*. Figure 1 on page 2 depicts some examples of algebraic varieties.

Singularities complicate the geometry of algebraic varieties. Intuitively, a singularity is a point of an algebraic variety at which the variety is pinched or folds onto itself. More formally, a variety is *nonsingular at a point* if the variety is diffeomorphic to Euclidean space around that point, and *singularities* are points at which a variety fails to be nonsingular. A variety is *singular* if it has a singularity and *nonsingular* otherwise. Figure 2 on page 3 depicts a singular curve called the *nodal cubic*.

A natural question in algebraic geometry asks whether any singular variety can be made nonsingular by surgical operations on the singular locus. If surgically removing singularities is possible, then we call any nonsingular variety so obtained a *resolution (of singularities)* of the original singular variety. Figure 3 on page 3 depicts a resolution of singularities of the nodal cubic. In a remarkable contribution, Hironaka [33, 34] proved the existence of a resolution of singularities for any algebraic variety defined over the complex numbers—or, more generally, over any field of characteristic 0. To be precise, Hironaka [33, 34] described a formal procedure for constructing a resolution of any singular variety. However, his results are silent on the geometry of other possible resolutions and how they relate to the one constructed by his procedure.

In this thesis, we consider the problem of analyzing the set of *all* possible resolutions of singularities of a singular algebraic variety. For curves and surfaces, it turns out that there is a unique "minimal" resolution of singularities that differs as little as possible from the starting singular variety. For threefolds (three-dimensional varieties), however, there is not in general a unique minimal resolution—there can be several "minimal" resolutions with different geometric properties.[2] Our contribu-

[1] It turns out that many spaces of solutions to systems of analytic equations are guaranteed to be algebraic varieties. Specifically, Chow [12] and Serre [73] have shown that varieties defined by systems of complex analytic equations in projective space are algebraic. Hence, the restriction to polynomial equations in defining algebraic varieties is sometimes innocuous.

[2] Following Mori [61]), we define "minimality" using numerical properties of the *canonical class*—a

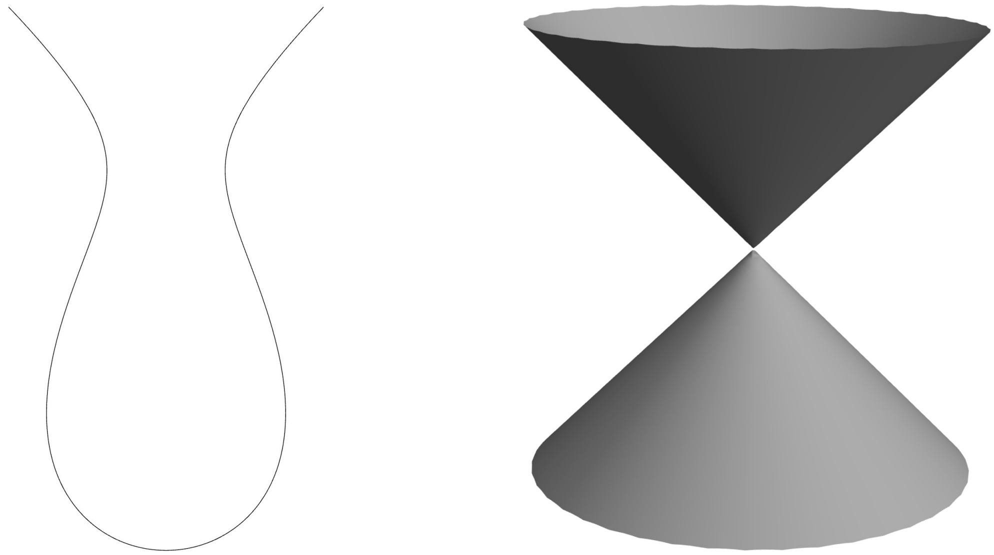

Figure 1: **Algebraic varieties** The left panel depicts an *elliptic curve*, which is the set of solution to a general cubic equation in two variables. The right panel depicts a *quadric cone*, which is the set of solutions to the three-variable equation $z^2 = x^2 + y^2$. The quadric cone is pinched at its apex. Hence, the apex is a *singularity* of the quadric cone.

tion is to shed some light on the structure of the set of all "minimal" resolutions of threefolds with simple singularities called *compound Du Val (cDV)* singularities.

Our analysis is motivated by geometric predictions from theoretical physics. The mathematical objects involved in F-theory are *elliptic fibrations*—which are algebraic varieties that consist of families of elliptic curves—and their associated *Weierstrass models*—which describe simple forms in which elliptic fibrations can appear. Places at which the elliptic curves in a family degenerate often give rise to cDV singularities of the associated Weierstrass model. Physicists have investigated the geometry of Weierstrass models, and have shown that physical dualities between F-theory and M-theory lead to geometric predictions regarding the structure of the set of all "minimal" resolutions of Weierstrass models.[3]

In this thesis, we provide mathematical proofs of some of the geometric predictions of F-theory. While our results have the flavor of F-theoretic predictions, we go beyond

mathematical object that is canonically associated to any sufficiently well-behaved variety.

[3]See Witten [84], Morrison and Vafa [66, 67], Vafa [79], Intriligator et al. [35], and Morrison and Seiberg [63] for seminal work in this vein.

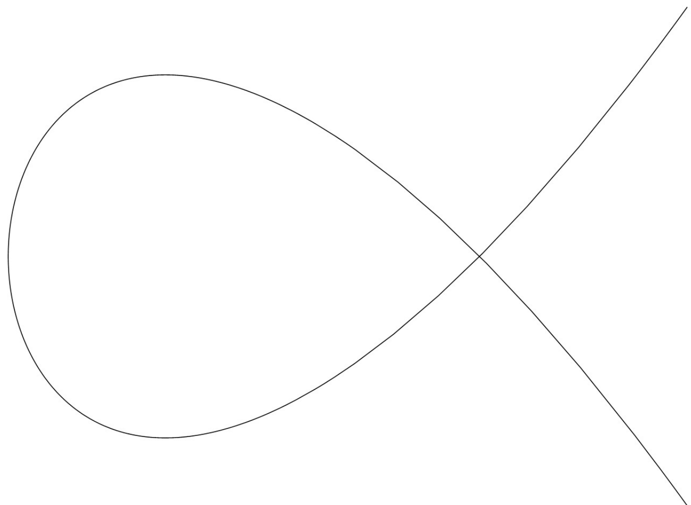

Figure 2: **The nodal cubic.** This curve is the set of solutions to the equation $y^2 = x^3 + x$. There is a singularity at the point at which the curve holds onto itself.

Figure 3: **Resolution of singularities of the nodal cubic.** The resolution (in orange) of the nodal cubic (in black) uses a third dimension to separate the branches of the nodal cubic at its singularity, using the mathematical process of *blowing up* the singularity. In the case of the nodal cubic, blowing up the singularity surgically replaces the singular point by two nonsingular points.

the scope of F-theory by considering threefolds with cDV singularities that are not the Weierstrass models of elliptic fibrations. Therefore, in addition to proving formal proofs of some of the geometric predictions of F-theory that were previously derived based on physical arguments, we show that the predictions of F-theory extend to some settings in which the physical arguments do not apply.

1 Introduction

In this thesis, we study the structure of the set of crepant resolutions of complex threefolds with $\mathbb{Q}$-factorial compound Du Val singularities. Because resolutions of singularities are birational morphisms, our analysis requires ideas and techniques from the theory birational geometry. To put our results into context, we first recall some ideas from the theories of birational geometry and singularities—with a focus on the issues that arise in dimension 3. We then describe the geometric predictions of F-theory—which are the inspiration for our work—and introduce our results.

1.1 Birational geometry

Birational equivalence provides a coarsening of the notion of isomorphism of algebraic varieties. Intuitively, two varieties are *birationally equivalent* if and only if they can be made isomorphic by removing (possible different) proper algebraic subvarieties—that is, if the varieties are generically isomorphic. Formally, we say that two varieties are birationally equivalent if there is a *birational map*—that is, a map that is generically an isomorphism—between them.

One of the basic questions in birational geometry asks how to obtain a member of a birational equivalence class of nonsingular varieties that is as "simple" as possible. This question can be interpreted as a step toward more difficult problem of classifying all (nonsingular) varieties up to isomorphism and understanding all birational maps between them. The formal definition of "as simple as possible"—which is due to Mori [61]—is subtle in dimensions 3 and higher. We therefore review the theory of birational geometry in dimensions 1 and 2 before turning to the case of higher-dimensional varieties.

In the case of curves, it is trivial to obtain the simplest member of a birational equivalence class. Indeed, any birational map between nonsingular, complete curves is an isomorphism. In particular, every nonsingular, complete curve is the unique "simplest" member of its birational class.

In the case of surfaces, there is a straightforward process for the simplification via birational transformations. Unlike in the case of curves, there are non-trivial birational transformations between nonsingular surfaces. For example, we can *blow up* a variety at a point—or, more generally along proper subvariety. In the case of nonsingular surfaces, blowing up at a point defines a surgery operation that replaces the point by an exceptional rational curve. The inverse of blowing up is called *contracting* the exceptional curve. A nonsingular surface is *minimal* if no rational curve on it can be contracted without introducing singularities. Given an arbitrary nonsingular surface, the strategy to obtain a birational minimal surface is to iteratively contract rational curves without introducing singularities—until no further rational curves can be contracted without introducing singularities. At each step, there may be several possible rational curves that can be contracted. However, the outcome of the process turns out to be independent of starting choices—as long as the starting surface is not birational to $C \times \mathbb{P}^1$ for an algebraic curve C. Indeed, there is exactly 1 minimal surface in each the birational equivalence class—except for the birational equivalence classes ruled surfaces.[4] Hence, similarly to the case of curves, there are unique minimal surfaces in essentially every birational equivalence class.

Birational geometry is much more subtle in dimensions 3 and higher. In this context, blowing up a nonsingular variety along any nonsingular subvariety surgically replaces the center of the blow-up by a codimension 1 subvariety instead of simply a rational curve. We call codimension 1 subvarieties *(prime) divisors*, as they are analogous to (codimension 1) prime ideals of commutative rings. In dimensions 3 and higher, we must therefore contract divisors instead of curves to invert the process of blowing up. We call a morphism that contracts a divisor a *divisorial contraction*.

The first complication of birational geometry in dimensions 3 and higher relates to problem of defining "minimality." The issue is that there are varieties that can be simplified by birational transformations but on which no divisor can be contracted. Specifically, there can be obstructions to contracting divisors that live in codimension 2. These obstructions can be removed by birational codimension 2 surgery operations called *flips*, so that performing a sequence of flips can allow further divisors to be contracted. As a result, in higher dimensions, it is not satisfactory to define

[4] This fact is proven, for example in [49, Chapter 1].

"minimality" as the inability to contract a divisor without introducing singularities.

In a remarkable contribution, Mori [61] proposed to define minimality in terms of the canonical class. The *canonical class* is a divisor that is canonically associated to every sufficiently well-behaved algebraic variety.[5] Mori [61] proposed to call a nonsingular threefold *minimal* if its canonical bundle is *numerically effective (nef)*—in that it has nonnegative intersection number with every curve on the variety. For most nonsingular surfaces, minimality in the sense of Mori is equivalent to the property tha no curve can be contracted without introducing singularities.[6]

For it to be possible to make the canonical class nef by performing birational transformations, the canonical class must initially be sufficiently positive. Specifically, all sufficiently large multiples of the canonical class must have nonzero sections—in which case we say that the variety has *nonnegative Kodaira dimension*.[7] If a variety does not have nonnegative Kodaira dimension, then the goal is to reduce the variety to a fibration of Fano varieties over a lower-dimensional base, as it is impossible to obtain a birational minimal model. We can then study the birational geometry of the original variety by studying the birational geometry of a general fiber and the birational geometry of the base of the fibration. For varieties of nonnegative Kodaira dimension, the goal is to obtain a birational minimal model.

The second complication of higher-dimensional birational geometry is that we may need to allow mild singularities to obtain a minimal model of a given variety, as Ueno [78] has shown. Specifically, divisorial contractions and flips can introduce $\mathbb{Q}$-*factorial*

[5]Formally, that the *canonical bundle* of a nonsingular variety is the invertible sheaf of top-dimensional differential forms. The *canonical class* is the first Chern class of the canonical bundle; more concretely, the canonical class is the class of the divisor of zeros and poles of a top-dimensional meromorphic differential form. The definition of the canonical class can be extended to normal varieties—see, for example, Section 3.3.

[6]To be precise, the equivalence holds for surfaces of nonneegative Kodaira dimension. To understand the equivalence, recall Castelnuovo's contractibility criterion [31, Theorem V.5.6], which asserts that a rational curve can be contracted without introducing singularities if and only if the curve has self-intersection number -1. By the adjunction formula [49, Proposition 5.73], it follows that a rational curve can be contracted without introducing singularities if and only if the intersection number of the canonical class with the curve is -1. Hence, if the canonical class of a surface is nef, then no curve can be contracted without introducing singularities. To show the converse, we apply the cone theorem on the structure of the cone of curves of a nonsingular variety [61, Theorem 1.3] to construct an "extremal" rational curve with which the canonical class has intersection number -1, -2, or -3. For surfaces of nonnegative Kodaira dimension, only the first case is possible [49, Theorem 1.28]. See Kollár and Mori [49, Chapter 1] for the details of the argument.

[7]In general, if the canonical ring is finitely generated, the Kodaira dimension is the dimension of the image of the pluricanonical map, which is only defined if the Kodaira dimension is positive. The Kodaira dimension is always a nonnegative integer or $-\infty$.

terminal singularities. Conversely, given any non-minimal threefold, it is possible to perform a divisorial contraction or a flip without introducing any singularities that are not $\mathbb{Q}$-factorial terminal. Intuitively, a singularity is $\mathbb{Q}$-factorial terminal if it is impossible to even partially resolve the singularity without moving the canonical class further from being numerically effective. In dimensions 1 and 2, there are no nontrivial terminal singularities, which explains why we can restrict to nonsingular varieties and still obtain the existence of minimal models in dimensions 1 and 2.

In an attempt to prove the existence of minimal models of all nonsingular varieties of nonnegative Kodaira dimension, Mori [62] described an iterative simplification process called the *minimal model program*, which can be applied in all dimensions. The procedure iteratively performs divisorial contractions or, when that is impossible, performs flips. At each step, the procedure seeks to modify the variety as little as possible. In dimension 3, Mori [62] showed that the minimal model program terminates, and hence reduces every nonsingular threefold of nonnegative Kodaira dimension (resp. negative Kodaira dimension) to a birational minimal model (resp. Fano fibration). In dimensions 4 and higher, the termination of the minimal model program turns out to be even more subtle than in dimension 3.[8]

There is also a third complication of higher-dimensional birational geometry: even if they exist, minimal models may not be unique. That is, there are often many minimal models in one birational class. To understand why, note that the numerical effectiveness of the canonical class is essentially determined in codimension 1. Hence, the geometry of birational minimal models can differ in codimension 2 and higher. For curves and (nonsingular) surfaces, there is no nontrivial geometry in codimension 2. In dimensions 3 and higher, however, there is non-trivial geometry in codimension 2 and higher that can vary between birational minimal models. Specifically, there are codimension 2 surgery operations called *flops*, which preserve the property of being a minimal model.[9] Flops are closely related to flips, which are the codimension 2 surgery operations that arise in the minimal model program. However, flips help to "simplify" a variety by changing the numerical intersection properties of the canonical class, while flops arise even between minimal models—which are already as "simple"

[8]Some results have been proven regarding the termination of flips—and hence the termination of the minimal model program—in dimension 4. See, for example, Kawamata et al. [41], Matsuki [56], Fujino [25], and Alexeev et al. [1].

[9]In dimension 3, Kollár [48] has shown that birational minimal models are related by finite sequences of (extremal) flops. See also Theorem 3.40 in Section 3.

as possible.

1.2 Resolution of singularities

Our results deal more closely with the theory of resolution of singularities, which is closely connected to—and shares many of the subtleties of—birational geometry. Intuitively, a *resolution of singularities* of a singular variety is a nonsingular variety that is obtained from the singular variety by surgery operations on its singular locus. More formally, a resolution of singularities of a variety is a proper, birational morphism from a nonsingular variety to the original variety that is an isomorphism away from the singular locus of the original variety. The first question in the theory of resolutions asks whether any singular variety admits a resolution of singularities. Generalizing work by Zariski [85, 86, 87], Hironaka [33, 34] answered this question in the affirmative for varieties of arbitrary dimension over fields of characteristic zero.

Given Hironaka's result, we can ask how singularities can be resolved in a way that modifies the variety as little as possible. The goal is then to avoid introducing any unnecessary divisors in the resolution—that is, to find a relative minimal model of a resolution over the starting singular variety. Formally, a morphism between varieties is a *relative minimal model* if the source has $\mathbb{Q}$-factorial terminal singularities and its canonical class is *relatively nef*—i.e., has nonnegative intersection number with any curve that lies in a single fiber of the morphism. We consider relative minimal models instead of absolute minimal models because the starting singular variety may itself have a canonical class that is not nef, and here our goal is to avoid modifying the starting singular variety more than is necessary to obtain a resolution. Providing a complete solution to the problem of finding a resolution that modifies a singular variety as little as possible requires classifying *all* relative minimal models of (projective) resolutions and understanding how the (relative) minimal models relate to one another. This classification problem is a question in (relative) birational geometry.

In the cases of curves and surfaces, the structure of the set of resolutions of singularities is simple. As nonsingular curves and nonsingular, (relatively) minimal surfaces are their unique relative minimal models, every singular curve or surface admits a unique "minimal" resolution of singularities,[10] which turns out to be universal among all resolutions of singularities.

[10]Singular curves actually admit unique resolution of singularities because every birational map between nonsingular, complete curves is an isomorphism.

In dimensions 3 and higher, studying the set of resolutions is more complicated due to the subtleties of higher-dimensional birational geometry. First, we must define minimality in terms of the canonical class instead of in terms of the inability to contract divisors without introducing singularities. Specifically, due to the possibility of flips occurring when running the (relative) minimal model program, there may be resolutions that are not relative minimal models which nevertheless do not have any divisor that can be contracted. Second, we should allow $\mathbb{Q}$-factorial terminal singularities, and therefore we should actually study a class of*partial resolutions* instead of full resolutions. Indeed, mild singularities may arise during the process of simplifying a resolution using the (relative) minimal model program.[11] Third, there may be multiple relative minimal models, so that there is no unique "minimal resolution." Specifically, resolutions may have multiple relative minimal models—which must be related by flops (at least in the case of threefolds). In particular, relative minimal models are in general not universal among all resolutions or among all partial resolutions with $\mathbb{Q}$-factorial terminal singularities.

To study of the set of relative minimal models of resolution in dimensions 3 and higher, we follow an approach taken by Brieskorn [9], Reid [71], and Matsuki [57]. The strategy is to analyze the relationships between the *movable cone* and the *nef cones*. Recall that an invertible sheaf is *movable* if it is (relatively) globally generated in codimension 1. The set of $\mathbb{R}_{>0}$-linear combinations of movable (resp. nef) invertible sheaves forms the movable (resp. nef) cone in cohomology. The KKMR decomposition theorem—which was proven by Kawamata [39], Kollár [48], Mori [61], and Reid [71]—shows that the movable cone of *any* (relative) minimal model decomposes canonically as the union of the nef cones of *all* (relative) minimal models. Moreover, the nef cones are locally polyhedral and disjoint in their interiors, and there is a geometric criterion for when the nef cones of two non-isomorphic (relative) minimal models share a codimension 1 face. Therefore, characterizing the KKMR decomposition provides geometric information regarding the set of relative minimal models and how they relate to one another.

[11] As terminal singularities can only arise in dimensions 3 and higher, they do not affect the existence of minimal resolutions arise in dimensions 1 and 2.

1.3 F-theory

In this thesis, we develop a Lie-theoretic characterization of the KKMR decompositions of relative minimal models of resolutions of a class of singular threefolds. Our characterization, which is based on ideas from the F-theory literature, provides geometric information regarding the set of all relative minimal models of resolutions. To motivate our analysis, we summarize some of the ideas of F-theory from a mathematical perspective.

F-theory has analyzed the geometry of elliptic n-folds for $2 \leq n \leq 5$. To illustrate the geometric predictions, we focus on the case of elliptic threefolds, which are the total spaces of elliptic fibrations over nonsingular surfaces. Formally, let B be a nonsingular surface. A *genus 1 fibration* is a flat morphism $\pi : X \to B$ from a nonsingular, complex projective threefold X to B whose generic fiber is a curve of genus 1 over the function field of B. An *elliptic fibration* consists of a genus 1 fibration $\pi : X \to B$ and a section $\sigma : B \to X$ whose image lies in the smooth locus of π. Given an elliptic fibration $\pi : X \to B$, we can associate a *Weierstrass model* W by analogy with embedding of an elliptic curve in the projective plane.[12]

When the total space X is Calabi–Yau, dualities between F-theory and M-theory in string theory predict that the decomposition of the "extended Kähler cone" into the "Kähler cones" of the relative minimal models of resolutions of W is determined by the structure of the Coulomb branch of a supersymmetric gauge theory (see Witten [84], Morrison and Vafa [66, 67], Vafa [79], Intriligator et al. [35], and Morrison and Seiberg [63]). The decomposition of the extended Kähler cone into the Kähler cones of relative minimal models of resolutions is the differential-geometric counterpart of the (algebro-geometric) KKMR decomposition. The Coulomb branch is determined by the geometry of X along the singular locus of the structure morphism π. Singular fibers of π over points of codimensions 1 and 2 in B correspond in M-theory to charged gauge fields and matter fields respectively [80]. The Intrilegator–Morrison–Seiberg [35] superpotential connects these data to the semisimple part of a gauge algebra and a representation of the gauge algebra—called the *matter representation*.[13]

We obtain a hyperplane arrangement in the dual fundamental Weyl chamber of the

[12]Formally, we define the Weierstrass model by $W = \mathrm{Proj}_B \bigoplus_{n=0}^{\infty} \pi_* \mathcal{O}_X(nS)$, where S is the image of the section σ. See, for example, Mumford and Suominen [68].

[13]When the elliptic fibration has rank 0, the gauge group is semisimple, as suggested by Mayrhofer et al. [59] and Morrison and Taylor [65].

Lie algebra of the gauge group from the hyperplanes that are normal to the weights of the matter representation—yielding a decomposition of the dual fundamental Weyl chamber into closed cones. Following Witten [84], the physics literature has loosely conjectured that this decomposition corresponds to the KKMR decomposition (or, equivalently, the decomposition of the extended Kähler cone into Kähler cones) of a relative minimal model of the elliptic fibration over its Weierstrass model—under an identification between the movable cone of a minimal model with the dual fundamental Weyl chamber.[14] This conjectural description has been verified in several low-rank examples by Esole et al. [19, 21–24] and Braun and Schäfer-Nameki [7, 8]. Esole et al. [19, 21–24] have also observed that the Calabi–Yau condition appears unnecessary provided that we pass to a relative minimal model of the elliptic fibration.[15]

From a mathematical perspective, the F-theory literature has associated a complex simple Lie algebra to each singular fiber in codimension 1 from the Kodaira [45–47] classification of the singular fibers of elliptic surfaces. To be precise, the Kodaira classification implies that the dual graph of each singular fiber is an affine Dynkin diagram. By removing the extra node, we obtain a standard Dynkin diagram and hence an isomorphism class of complex simple Lie algebras. These simple Lie algebras are called the *gauge factors* in F-theory. The semisimple part of the gauge algebra is obtained by multiplying the gauge factors associated to all codimension 1 singular fibers. On the other hand, the F-theory literature has not given a precise, mathematical definition of the matter representation in general.[16] Morrison and Taylor [64] have defined the matter representation in several cases with simple gauge algebras from the intersection numbers of exceptional curves with exceptional divisors and have suggested that similar methods might work in general.

1.4 This thesis

We focus on a particularly simple type of threefold singularities—the *compound Du Val (cDV) singularities*—to obtain a general characterization of the KKMR decomposition for relative minimal models of a resolutions. The characterization, which

[14]The combinatorics of this decomposition of the dual fundamental Weyl chamber has been studied by Hayashi et al. [32] and Esole et al. [18, 20].

[15]As Calabi–Yau manifolds have trivial canonical classes, they are minimal models. Hence, we do not need to pass to a relative minimal model if the starting elliptic fibration is Calabi–Yau.

[16]However, general heuristics/predictions have been given using anomaly cancellation for the Intrilegator–Morrison–Seiberg superpotential [35, 63]. See also Grassi and Morrison [27, 28].

is novel to this thesis, resembles the conjectural description of the KKMR decompositions of the relative minimal models of resolutions of the Weierstrass models of elliptic threefolds from the F-theory literature.

The class of singularities that we consider—the cDV singularities—are the singularities through which there are surface sections with Du Val singularities. For threefolds, cDV singularities are the only singularities that admit *small* resolutions—that is, resolutions all of whose fibers are points or (possibly non-integral) curves—as Reid [71] has shown. The restriction to cDV singularities therefore allows us to focus on the geometry of exceptional curves and exceptional families of curves in our analysis. Because the class of threefolds with cDV singularities includes the Weierstrass models of elliptic threefolds,[17] our starting point goes beyond the scope of the F-theory literature—more than simply relaxing the Calabi–Yau condition as Esole et al. [19, 21–24] have done.

For cDV singularities, there is a particularly simple characterization of the resolutions that are minimal models: a (projective) resolution is a minimal model if and only if the resolution does not change the canonical class.[18] Such resolutions are said to be *crepant*.[19] Intuitively, for sufficiently simple singularities, we do not need to modify the canonical class so that it has strictly positive intersection number with any exceptional curve to obtain a resolution. As a result, (projective) crepant partial resolutions with $\mathbb{Q}$-factorial terminal singularities can be obtained for threefolds with compound Du Val singularities.[20] Conversely, any crepant partial resolution with $\mathbb{Q}$-factorial terminal singularities is by definition a relative minimal model over the starting singular variety.

Starting with a threefold with cDV singularities, we obtain a simple factor of the gauge algebra for each codimension 2 point of the singular locus. Intuitively, a general hyperplane section through a general point of each curve in the singular locus has a Du Val singularity. Considering the dual graph of a minimal resolution, we

[17]To see why the Weierstrass models of elliptic threefolds have cDV singularities, note that (flat, nonsingular) elliptic fibrations provide small resolutions of their Weierstrass models by construction.

[18]This property is even true for the more general class of *canonical* singularities, which are the possible singularities of (relative) canonical models.

[19]The change in the canonical class is referred to as the *discrepancy*. Reid [71] coined the terminology crepant to mean "not discrepant".

[20]This property applies more generally to surfaces and threefolds with canonical singularities. Indeed, in this case, we can take any resolution of singularities and run the relative minimal model program to obtain a crepant partial resolution with $\mathbb{Q}$-factorial terminal singularities.

obtain a simply-laced connected Dynkin diagram due to the results of Du Val [13]. Taking monodromy into account, we can obtain non-simply-laced Dynkin diagrams, as shown by Lipman [54] and Esole et al. [22, 23]. The Dynkin diagrams give rise to simple Lie aglebras for all codimension 2 points of the singular locus. Multiplying the simple Lie algebras associated to all codimension 2 singular points, we obtain a gauge algebra. When the starting singular threefold is $\mathbb{Q}$-factorial,[21] we show that there is an identification between a Cartan subalgebra and a cohomology space under which the dual fundamental Weyl chamber coincides with movable cones of the relative minimal models of resolutions. We also construct "matter representations" for each minimal model, which we show are closely related to the ample cones. Our combinatorial description of the matter representation sheds light on the set of possible codimension 1 faces of nef cones, and allows us to deduce an effective bound on the number of crepant partial resolutions of $\mathbb{Q}$-factorial threefolds with cDV singularities.

Motivated by the approaches of Reid [71, 72], Mori [61], Katz and Morrison [36], and Cattaneo [11], our definition of the matter representation at a singular point relies on considering a general hyperplane section through the singular point. Such a hyperplane section has a Du Val singularity, and Reid [71] has shown that any crepant partial resolution of the starting threefold gives rise to a crepant partial resolution of the hyperplane section. To obtain a matter representation, we consider the intersection numbers of divisors with not only between the exceptional curves but also 1-cycles that correspond to other roots in the root system with the same ADE type as the hyperplane section's Du Val singularity. This definition of the matter representation is crucial to our main theorem, which provides conditions under which the matter representation is independent of the choice of crepant resolution of the starting singular threefold. As a consequence, we show that the KKMR decomposition coincides with the decomposition of the dual fundamental Weyl chamber of the gauge algebra into closed chambers for the hyperplane arrangement consisting of the hyperplanes that are normal to the weights of the matter representation.

Our characterization of the KKMR decomposition is closely related to previous work by Brieskorn [9] and Matsuki [57]. Brieskorn [9] has characterized the KKMR decomposition of the relative minimal models of resolutions of families of Du Val

[21] $\mathbb{Q}$-factoriality requires that some multiple of every codimension 1 subvariety is cut out by a single equation. This condition is analogous to the requirement in the physics literature that the elliptic fibration have rank 0 for its gauge algebra to be semisimple. Indeed, if an elliptic fibration has rank 0, then its Weierstrass model is $\mathbb{Q}$-factorial, as we show in Proposition 8.6 in Section 8.

singularities. The total spaces of such families have cDV singularities, and hence our starting point is more general than that of Brieskorn [9]. However, our KKMR decomposition result does not generalize that of Brieskorn [9] because we impose stronger auxiliary conditions. Matsuki [57] has proven a form of the conjectural description of the KKMR decomposition for the crepant resolutions of Weierstrass models of elliptic threefolds. Crucially to his argument, Matsuki [57] has assumed that the discriminant locus of the elliptic fibration has simple normal crossings and that only Kodaira fibers appear in codimension 2. However, non-Kodaira fibers can appear in codimension 2 in general—as shown by Miranda [60], Lawrie and Schäfer-Nameki [51], Braun and Schäfer-Nameki [7, 8], and Esole et al. [22, 23, 24] in many examples—and the discriminant in general does not have simple normal crossings—as observed by Esole and Yau [17] and Lawrie and Schäfer-Nameki [51]. Our hypotheses are different than those of Matsuki [57], but we allow for non-Kodaira fibers as well as for more general discriminant loci. We illustrate the use of these additional degrees of generality in our examples.

In general, we hope that the framework proposed in this thesis may provide a useful starting point for applying F-theoretic intuition to understanding the structure of the set of relative minimal models of resolutions. It would be interesting to determine the extent to which the story developed in this thesis can be extended to settings in which the main hypotheses—namely that the starting singular variety is three-dimensional, $\mathbb{Q}$-factorial, and has cDV singularities—are not satisfied.

1.5 Outline of this thesis

The remainder of this thesis is organized as follows.

In Sections 2 and 3, we review background material. Specifically, in Section 2, we briefly reviews basic facts from the theory of root systems. In Section 3, we review relevant facts from intersection theory, birational geometry, and surface and threefold singularities. Our presentation in Section 3 is guided by illustrative examples.

Sections 4–8 comprise the heart of this thesis and consist of original material. In Section 4, we construct the gauge algebra and the matter representations. In Section 5, we formally state our main results and provide basic applications. In Sections 6 and 7, we prove results that we assert in Section 4. In Section 8, we present a family of examples from the Weierstrass models of (possibly singular) elliptic fibrations with non-Kodaira fibers in codimension 2.

Appendix A presents a standard proof that we omit from Section 3.

2 Root systems

In this section, we briefly review elements of the theory of root systems. Our treatment follows Kirillov [42].

To fix terminology, we call a finite-dimensional real inner product space a *Euclidean vector space*. We denote the inner product by $\langle -, - \rangle$. Given a Euclidean vector space V and a nonzero vector $\alpha \in V$, let $s_\alpha : V \to V$ denote the orthogonal operator given by reflection through the hyperplane perpendicular to α.

A *root system* in a Euclidean vector space V is a set $\mathfrak{R} \subseteq V \setminus \{0\}$ that generates V as a vector space, such that:

- for all $\alpha, \beta \in V$, we have that

$$\frac{2\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} \in \mathbb{Z} \quad \text{and} \quad s_\alpha(\beta) \in \mathfrak{R};$$

- for all $\alpha \in \mathfrak{R}$, the only multiples of α that lie in $\mathfrak{R}$ are $\pm\alpha$.

Thus, we require root systems to be reduced and crystallographic. An *isomorphism* from a root system $\mathfrak{R}$ in V to a root system $\mathfrak{R}'$ in V' is an orthogonal isomorphism $\theta : V \to V'$ such that $\theta(\mathfrak{R}) = \mathfrak{R}'$. Given root systems $\mathfrak{R}$ in V and $\mathfrak{R}'$ in V', the set $\mathfrak{R} \cup \mathfrak{R}'$ is a root system in $V \oplus V'$ and that we call the *direct sum* $\mathfrak{R} \oplus \mathfrak{R}'$ of $\mathfrak{R}$ and $\mathfrak{R}'$. Here, we are abusing notation and writing $\mathfrak{R}$ for $\{(\alpha, 0) \,|\, \alpha \in \mathfrak{R}\} \subseteq V \oplus V'$ to regard $\mathfrak{R}$ as a subset of $V \oplus V'$, and similarly writing $\mathfrak{R}'$ for $\{(0, \alpha) \,|\, \alpha \in \mathfrak{R}'\} \subseteq V \oplus V'$ to regard $\mathfrak{R}'$ as a subset of $V \oplus V'$. A root system is *reducible* if it is isomorphic to a nontrivial direct sum of root systems and *irreducible* otherwise.

Henceforth, we require that $\mathfrak{R} = \emptyset$ or that $\min_{\alpha \in \mathfrak{R}'} \langle \alpha, \alpha \rangle = 2$ for every root subsystem $\mathfrak{R}' \subseteq \mathfrak{R}$ that appears in a direct sum decomposition of $\mathfrak{R}$. This normalization of the inner product comes from a normalized Killing form of the corresponding semisimple Lie algebra. A root system $\mathfrak{R}$ is *simply-laced* if $\langle \alpha, \alpha \rangle = 2$ for all $\alpha \in \mathfrak{R}$.

A *polarization* of a root system $\mathfrak{R}$ in a vector space E is a functional $\lambda \in V^*$ such that $0 \notin \lambda(\mathfrak{R})$. Given a polarization λ, a root α is *positive (resp. negative) with respect to* λ if $\lambda(\alpha) > 0$ (resp. $\lambda(\alpha) < 0$). A positive root is *simple with respect to*

λ if it cannot be expressed as a nontrivial $\mathbb{Z}_{\geq 0}$-linear combination of positive roots. The *root lattice* is the lattice $\mathfrak{L}(\mathfrak{R})$ in V generated by the roots. It turns out that the simple roots form a $\mathbb{Z}$-basis for the root lattice and that the positive roots are the roots that are nonnegative integral combinations of simple roots.

Proposition 2.1 ([42, Corollary 7.18])**.** *The set of simple roots (with respect to any polarization) forms a $\mathbb{Z}$-basis for the root lattice, and a root is positive (resp. negative) if and only if it is $\mathbb{Z}_{\geq 0}$-linear (resp. $\mathbb{Z}_{\leq 0}$-linear) combination of simple roots.*

The *Killing matrix (with respect to λ)* is the matrix whose entries are $\langle \alpha_i, \alpha_j \rangle$, where $\alpha_1, \ldots, \alpha_n$ are the simple roots. This matrix is the expansion of the normalized Killing form in a basis of simple roots. For simply-laced root systems, the Killing matrix coincides with the Cartan matrix, but for non-simply-laced root systems, the Killing matrix is symmetric while the Cartan matrix is not. Up to simultaneous permutation of its rows and columns, the Killing matrix is independent of the choice of polarization [42, Theorem 7.48]. Hence, we call the Killing matrix with respect to any polarization the *Killing matrix* of the root system. The Killing matrices of irreducible root systems have been classified.

Theorem 2.2 (Classification of irreducible root systems [42, Theorem 7.49])**.** *Table 1 on page 17 lists the possible Killing matrices of irreducible root systems up to simultaneous permutation of rows and columns.*

We can use Theorem 2.2 to produce root systems from configurations of vectors in a lattice whose matrix of inner products is one of the possible Killing matrices of irreducible root systems.

Corollary 2.3. *Let L be a lattice with form $\langle -, - \rangle$, and let $\Delta \subseteq L$. If the form $\langle -, - \rangle$ expands (with respect to the set Δ) to a Killing matrix of Table 1 on page 17, then there is a unique root system $\mathfrak{R} \subseteq L$ in $\operatorname{span}_{\mathbb{R}} \Delta \subseteq L \otimes \mathbb{R}$ of which Δ is the set of simple roots.*

Proof. It follows from Theorem 2.2 that there is a root system $\mathfrak{R}$ in $\mathfrak{L} \otimes \mathbb{R}$ with set of simple roots Δ. Proposition 2.1 implies that $\mathfrak{L}(\mathfrak{R}) \subseteq \mathfrak{L}$, and hence we must have that $\mathfrak{R} \subseteq \mathfrak{L}$. Uniqueness follows from the fact that a root system is uniquely determined by its set of simple roots [42, Corollary 7.33]. □

Short elements of root lattices of simply-laced irreducible root systems are roots.

$$A_n : \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{bmatrix} \qquad B_n : \begin{bmatrix} 4 & -2 & 0 & \cdots & 0 & 0 \\ -2 & 4 & -2 & \cdots & 0 & 0 \\ 0 & -2 & 4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \vdots & 4 & -2 \\ 0 & 0 & 0 & \vdots & -2 & 2 \end{bmatrix}$$

$$C_n : \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \vdots & 2 & -2 \\ 0 & 0 & 0 & \vdots & -2 & 4 \end{bmatrix} \qquad D_n : \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \vdots & 2 & -1 & -1 \\ 0 & 0 & 0 & \vdots & -1 & 2 & 0 \\ 0 & 0 & 0 & \vdots & -1 & 0 & 2 \end{bmatrix}$$

$$F_4 : \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ 0 & 0 & -2 & 4 \end{bmatrix} \qquad E_n : \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \vdots & 2 & -1 & -1 & 0 \\ 0 & 0 & 0 & \vdots & -1 & 2 & 0 & -1 \\ 0 & 0 & 0 & \vdots & -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & \vdots & 0 & -1 & 0 & 2 \end{bmatrix}$$

$$G_2 : \begin{bmatrix} 2 & -3 \\ -3 & 6 \end{bmatrix}$$

Table 1: **Killing matrices of irreducible root systems.** The subscript on the type denotes the dimension of the ambient vector space. Type E_n is possible for and only for $n = 6, 7, 8$.

Proposition 2.4 (Folk result)**.** *If* $\mathfrak{R}$ *is a simply-laced irreducible root system and* $v \in \mathfrak{L}(\mathfrak{R})$ *satisfied* $\langle v, v\rangle = 2$*, then* $v \in \mathfrak{R}$*.*

Proof sketch. Note that each of the root systems A_n, D_n, E_6, E_7, and E_8 is defined as the set of length-$\sqrt{2}$ vectors in a lattice. By construction, this ambient lattice must contain the root lattice, and the proposition follows by Theorem 2.2. □

The *Weyl group* $\mathfrak{W}(\mathfrak{R})$ of a root system $\mathfrak{R}$ in a vector space V is the subgroup of $\mathrm{Aut}(V)$ generated by the reflections s_α for $\alpha \in \mathfrak{R}$. As $\mathfrak{W}(\mathfrak{R})$ acts naturally on V via orthogonal transformations that send roots to roots, $\mathfrak{W}(\mathfrak{R})$ acts naturally on $\mathfrak{L}(\mathfrak{R})$.

3 Geometric preliminaries

In this section, we review background material from intersection theory, the birational geometry of algebraic varieties, and the theories of Du Val and compound Du Val singularities. We illustrate the theories through the examples of the quadric cone and the conifold.

3.1 Intersection multiplicities and products

Given a Noetherian, integral scheme X, let $\mathrm{Pic}\, X$ denote the *Picard group* of isomorphism classes of invertible sheaves on X. We denote by $\mathrm{Pic}\, X_{/\mathrm{tors}}$ the Picard group modulo torsion. A *prime divisor* on X is a codimension 1 integral subscheme. A *divisor* (resp. $\mathbb{Q}$-*divisor*) is a $\mathbb{Z}$-linear (resp. $\mathbb{Q}$-linear) combination of prime divisors. We denote by $[Z]$ the divisor class of a prime divisor Z. Let $\mathrm{Cl}\, X$ denote the divisor class group of X.

Let $c_1 : \mathrm{Pic}\, X \to \mathrm{Cl}\, X$ denote the first Chern class homomorphism, which is injective if X is normal. If X is normal and c_1 is surjective (resp. has torsion cokernel), then we say that X is *factorial* (resp. $\mathbb{Q}$-*factorial*). For a normal, integral scheme X, a class $D \in \mathrm{Cl}\, X$ is *Cartier* (resp. $\mathbb{Q}$-*Cartier*) if some D (resp. some positive integral multiple of D) lies in $c_1(\mathrm{Pic}\, X)$. We can similarly define $\mathbb{Q}$-*Cartier divisors* and $\mathbb{Q}$-*Cartier* $\mathbb{Q}$-*divisors.* Note that a normal, integral scheme X is factorial (resp. $\mathbb{Q}$-factorial) if and only if every divisor is Cartier (resp. $\mathbb{Q}$-Cartier).

Example 3.1 (Singularities and factoriality). The Auslander–Buchsbaum Theorem [14, Theorem 19.19] guarantees that any regular scheme is factorial. On the other hand, singularities can (but do not always) cause a failure of $\mathbb{Q}$-factoriality or factoriality. For example, the *quadric cone* $Q = V(z^2 - xy) \subseteq \mathbb{P}^3_{\mathbb{C}}$ has a singularity at $x = y = z = 0$, and the prime divisor $V(x, z)$ is not Cartier. (The equation $z^2 = xy$ is related to the equation $z^2 = x^2 + y^2$—which defines the quadric cone depicted in Figure 1 on page 2—by a $\mathbb{C}$-linear change of coordinates.) However, Q is $\mathbb{Q}$-factorial. For example, we have that $2[V(x, z)] = c_1(\mathcal{O}_Q(1))$ in $\operatorname{Cl} X$ because the section x cuts out $2[V(x, z)]$ as a divisor. The *conifold* $X = V(zw - xy) \subseteq \mathbb{P}^4_{\mathbb{C}}$ is not even $\mathbb{Q}$-factorial, as the prime divisor $V(x, z)$ is not $\mathbb{Q}$-Cartier. (Note that the quadric cone and the conifold are cones over nonsingular quadrics in $\mathbb{P}^2$ and $\mathbb{P}^3$, respectively.)

If $f : Y \to X$ is a morphism between normal, integral schemes and D is a Cartier divisor on X, then we write f^*D for the pullback of D to Y, which is a Cartier divisor on Y. If D is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X, then we write $f^*D = \frac{1}{n}c_1(f^*\mathcal{L})$, where $nD = c_1(\mathcal{L})$. In this case, f^*D is well-defined as an element of $\operatorname{Cl} Y \otimes \mathbb{Q}$.

By a *variety*, we mean an irreducible, reduced, separated scheme of finite type over $\mathbb{C}$. Given a proper curve or 1-cycle C on a variety X and an invertible sheaf $\mathcal{L} \in \operatorname{Pic} X$, we define the intersection number of C with $\mathcal{L}$ by

$$\mathcal{L} \cdot C = \deg(\mathcal{L} \cap C) \in \mathbb{Z}.$$

As degrees are constant in flat families [81, Corollary 24.7.3], the intersection number depends only on the algebraic equivalence class of C. When X is normal, we can naturally extend the intersection pairing $- \cdot -$ to take $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors in the first argument, in which case the intersection number has values in $\mathbb{Q}$ instead of $\mathbb{Z}$. Intersection numbers with non-$\mathbb{Q}$-Cartier divisors are in general not well-defined.

Example 3.2 (Intersection numbers on non-factorial varieties [16, Examples 2.21 and 2.22]). Consider the varieties described in Example 3.1. As the quadric cone $Q = V(z^2 - xy) \subseteq \mathbb{P}^3$ is $\mathbb{Q}$-factorial, curves have well-defined intersection numbers with all divisors. However, these intersection numbers may not be integral because Q is not factorial. For example, we have that $[V(x, z)] \cdot V(x, z) = \frac{1}{2}$ on X. Indeed, as $V(x^2, xz, z^2) = V(x^2, xz, xy)$, the divisor $2[V(x, z)]$ is Cartier and is cut out by x. Hence, we have that $[V(x, z)] = \frac{1}{2}c_1(\mathcal{O}_X(1))$, so that

$$[V(x, z)] \cdot V(x, z) = \frac{1}{2}\mathcal{O}_X(1) \cdot V(x, z) = \frac{1}{2},$$

where the second equality uses the projection formula for intersection multiplicities [26, Proposition 2.5(c)] and Bézout's Theorem [26, Proposition 8.4].

On the other hand, rational (and hence algebraic) equivalence classes of curves do not have well-defined intersection numbers with divisors that are not $\mathbb{Q}$-Cartier. On the conifold $X = V(zw - xy) \subseteq \mathbb{P}^4$, for example, the intersection number $[V(x,z)] \cdot V(y,z,w)$ is not well-defined. Intuitively, we should have that $[V(x,z)] \cdot V(y,z,w) > 0$ because $V(y,z,w)$ and $V(x,z)$ meet transversely (at $[0;0;0;0;1]$). However, note that $V(y,z,w)$ is rationally equivalent to $V(y,w,u)$—where u cuts out the hyperplane at ∞ in $\mathbb{P}^4$—and $V(y,w,u) \cap V(x,z) = \emptyset$. Hence, it would appear that we should also have that $[V(x,z)] \cdot V(y,z,w) = 0$.

3.2 The cone of curves and classes of divisors

Our treatment of the cone of curves and related classes of divisors (ample, nef, and movable) follows Kollár and Mori [49] and Matsuki [58]. Throughout, we work in a relative setting as a setup for developing relative birational geometry.

Let $f : Y \to X$ be a morphism between varieties. We write $Z_1(Y/X)$ for the group of relative 1-cycles on Y—that is, the abelian group freely generated by the classes of curves in Y that map to points under f. We say that 1-cycles $C_1, C_2 \in Z_1(Y/X)$ are *numerically equivalent* if $\mathcal{L} \cdot C_1 = \mathcal{L} \cdot C_2$ for all $\mathcal{L} \in \operatorname{Pic} Y$. Let $N_1(Y/X) = (Z_1(Y/X)/\sim) \otimes \mathbb{R}$, where $\sim$ denotes the relation of numerical equivalence over X. Let $\operatorname{NE}(Y/X)$ denote the cone spanned in $N_1(Y/X)$ by the classes of effective 1-cycles, and let $\overline{\operatorname{NE}}(Y/X)$ denote the closure of $\operatorname{NE}(Y/X)$ in $N_1(Y/X)$.

We say that invertible sheaves $\mathcal{L}, \mathcal{L}' \in \operatorname{Pic} Y$, are *numerically equivalent over* X if $\mathcal{L} \cdot C = \mathcal{L}' \cdot C$ for all $C \in Z_1(Y/X)$. Let $\operatorname{Num} Y/X$ denote the group of invertible sheaves on Y modulo numerical equivalence over X. Define $N^1(Y/X) = \operatorname{Num} Y/X \otimes_{\mathbb{Z}} \mathbb{R}$. Note that the intersection pairing extends to a canonical duality between $N_1(Y/X)$ and $N^1(Y/X)$.

If $\mathcal{L}$ is numerically equivalent to $\mathcal{O}_Y$ over X, then we say that $\mathcal{L}$ is *numerically f-trivial*. By abuse of terminology, we extend the definition of numerical triviality to $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors on normal varieties. An invertible sheaf $\mathcal{L} \in \operatorname{Pic} Y$ is *f-ample* if there exists $n \in \mathbb{Z}_{>0}$ such that the complete relative linear series $(\mathcal{L}^{\otimes n}, f_*\mathcal{L}^{\otimes n})$ is basepoint free and the resulting morphism $Y \to \operatorname{Proj}_X \bigoplus_{i=0}^{\infty} (f_*\mathcal{L}^{\otimes n})^{\otimes i}$ is a closed embedding. An invertible sheaf $\mathcal{L} \in \operatorname{Pic} Y$ is *f-nef* if $\mathcal{L} \cdot C \geq 0$ for all $C \in \operatorname{NE}(Y/X)$

(or, equivalently, all $C \in \overline{\mathrm{NE}}(Y/X)$). An invertible sheaf $\mathcal{L} \in \operatorname{Pic} Y$ is *f-movable* if $f_*\mathcal{L} \neq 0$ and the support of the cokernel of the natural homomorphism $f^*f_*\mathcal{L} \to \mathcal{L}$ has codimension 2 or higher.

Let $\mathrm{Amp}(Y/X)$ (resp. $\overline{\mathrm{Amp}}(Y/X)$, $\mathrm{Mov}(Y/X)$) denote the cone in $N^1(Y/X)$ generated by the classes of f-ample (resp. f-nef, f-movable) invertible sheaves, and let $\overline{\mathrm{Mov}}(Y/X)$ denote the closure of $\mathrm{Mov}(Y/X)$. Because ample invertible sheaves are movable, we have that $\mathrm{Amp}(Y/X) \subseteq \mathrm{Mov}(Y/X)$. We say that a class $D \in N^1(Y/X)$ is *f-ample* (resp. *f-nef*, *f-movable*) if $D \in \mathrm{Amp}(Y/X)$ (resp., $D \in \overline{\mathrm{Amp}}(Y/X)$, $D \in \mathrm{Mov}(Y/X)$) by abuse of terminology.[22]

An important characterization of the ample cone is Kleiman's criterion [43].

Theorem 3.3 (Relative Kleiman criterion [49, Theorem 1.44])**.** *Let $f : Y \to X$ be a projective morphism between varieties. An invertible sheaf $\mathcal{L} \in \operatorname{Pic} Y$ is f-ample if and only if $\mathcal{L} \cdot C > 0$ for all $C \in \overline{\mathrm{NE}}(Y/X) \setminus \{0\}$.*

Hence, the ample cone is the interior of the nef cone [53, Theorem 1.4.23].

We now illustrate the preceding theory through two examples from resolutions of the quadric cone and the conifold. Recall that a proper, birational morphism $f : Y \to W$ between integral schemes is a *partial resolution* if Y is normal and f is an isomorphism above the regular locus of W. A partial resolution $f : Y \to W$ is a *resolution* if Y is regular.

In the examples, we use blow-ups to construct resolutions. Recall that the *blow-up* of a scheme X along a closed subscheme Z is

$$\mathrm{Bl}_Z X = \mathrm{Proj}_X \bigoplus_{n=0}^{\infty} \mathcal{I}_Z^n,$$

where $\mathcal{I}_Z$ is the quasicoherent sheaf of ideals that corresponds to Z.

Example 3.4 (Divisors on a resolution of the quadric cone)*.* As in Example 3.1, consider the quadric cone $Q = V(z^2 - xy) \subseteq \mathbb{P}^3$. Let $\widetilde{Q} = \mathrm{Bl}_{V(x,z)}$ denote the blow-up of Q along the line $V(x, z)$, and let $g : \widetilde{Q} \to Q$ denote the projection, which is a resolution of singularities. Let E denote the exceptional locus of f, which is a rational curve.

[22]The relative Kleiman criterion implies that the numerical class of any f-ample invertible sheaf consists of f-ample sheaves. It is clear that the numerical class of any f-nef invertible sheaf consists of f-nef sheaves. However, it is possible for f-movable and f-immovable invertible sheaves to be numerically equivalent over X [39, §2]. Nevertheless, if $\mathcal{L}$ and $\mathcal{L}'$ are numerically equivalent over X and $\mathcal{L}$ is f-movable, then some tensor power of $\mathcal{L}'$ is f-movable—see Kawamata [39, Lemma 2.2].

The self-intersection number of E is -2. Because the invertible sheaf $c_1^{-1}([E])$ generates the relative Picard group $\operatorname{Pic} \widetilde{Q}/f^* \operatorname{Pic} Q$, the class $[E]$ generates both $N^1(\widetilde{Q}/Q)$ and $N_1(\widetilde{Q}/Q)$. By definition, the cone of curves is $\operatorname{NE}(\widetilde{Q}/Q) = \overline{\operatorname{NE}}(\widetilde{Q}/Q) = \mathbb{R}_{\geq 0}[E]$. As E has negative self-intersection number, the movable cone and the nef cone are both the closed cone $\mathbb{R}_{\leq 0}[E]$ while the ample cone is the interior $\mathbb{R}_{<0}[E]$ by Theorem 3.3.

Example 3.4 illustrates a general point regarding the relationship between the movable cone and the nef cone: when Y is a surface and $f : Y \to X$ is a projective morphism, we have that $\overline{\operatorname{Mov}}(Y/X) = \overline{\operatorname{Amp}}(Y/X)$ [57, Remark II-5(2)]. Intuitively, movability is determined in codimension 1 while the property of being nef is determined in dimension 1. Codimension 1 and dimension 1 coincide in dimension 2 but not in higher dimensions. Indeed, in dimensions 3 and higher, we only have the inclusion $\overline{\operatorname{Mov}}(Y/X) \supseteq \overline{\operatorname{Amp}}(Y/X)$ in general.

Example 3.5 (Divisors on a small resolution of the conifold). As in Example 3.1, consider the conifold $X = V(zw - xy) \subseteq \mathbb{P}^4$. Let $Y = \operatorname{Bl}_{V(x,z)} X$ denote the blow-up of X along the line $V(x, z)$, and let $f : Y \to X$ denote the projection. Let C denote the exceptional locus of f, which is a rational curve. Let $D = f^{-1}(V(x, z))$ denote the prime divisor above the center of the blow-up, which is Cartier.

The intersection number $[D] \cdot C$ is -2. Hence, the class $[C]$ generates $N_1(Y/X)$. By definition, the cone of curves is $\operatorname{NE}(\widetilde{Q}/Q) = \overline{\operatorname{NE}}(\widetilde{Q}/Q) = \mathbb{R}_{\geq 0}[C]$. As the invertible sheaf $c_1^{-1}([D])$ generates the relative Picard group $\operatorname{Pic} \widetilde{Q}/f^* \operatorname{Pic} Q$, the class $[D]$ generates $N^1(Y/X)$. Moreover, the nef cone is the closed cone $\mathbb{R}_{\leq 0}[D]$, while the ample cone is the interior $\mathbb{R}_{<0}[D]$. On the other hand, as g is an isomorphism in codimension 1, every invertible sheaf is movable. Hence, the movable cone is $\operatorname{Mov}(Y/X) = N^1(Y/X) = \mathbb{R}[D]$.

3.3 Canonical classes and dualizing sheaves

Our treatment of canonical classes follows Kollár and Mori [49]. Let X be a normal variety, let X_{sing} denote the singular locus of X, and let $X_{\text{sm}} = X \setminus X_{\text{sing}}$. Let $\Omega_{X_{\text{sing}}} \in \operatorname{Pic} X_{\text{sm}}$ denote the *canonical bundle* of X_{sm}, which is the top exterior power of the cotangent bundle of X. Define the *canonical class* of X by $K_X = c_1(\Omega_{X_{\text{sm}}})$, where we identify $\operatorname{Cl} X = \operatorname{Cl} X_{\text{sm}}$ as $\operatorname{codim} X_{\text{sing}} \geq 2$. A normal variety X is $\mathbb{Q}$-*Gorenstein* if K_X is $\mathbb{Q}$-Cartier. Intuitively, $\mathbb{Q}$-Gorenstein varieties are the varieties for which intersection numbers with the canonical class are well-defined. A birational morphism $f : Y \to W$ is *crepant* if Y is normal, W is $\mathbb{Q}$-Gorenstein, and $f^*K_W = K_Y$ as elements of $\operatorname{Cl} Y \otimes \mathbb{Q}$. Intuitively, crepant morphisms do not change the canonical class—up to torsion in the divisor class group.

We next state a criterion for blow-ups to be crepant.

Proposition 3.6 (Folk result)**.** *Let X' be a nonsingular, quasi-projective variety and let $X = V(f)$ be a prime divisor on X. Let $Z \subsetneq X$ be a nonsingular proper subvariety. If X and $\operatorname{Bl}_Z X$ are normal, then the projection $\pi : \operatorname{Bl}_Z X \to X$ is crepant if and only if f vanishes to order exactly* $\operatorname{codim}_X Z - 1$ *at the generic point of Z.*

Proposition 3.6 has implications for the crepancy of resolutions, for example of the quadric cone and the conifold.

Example 3.7 (Crepant resolutions of the quadric cone and the conifold). As in Example 3.1, consider the quadric cone $Q = V(z^2 - xy) \subseteq \mathbb{P}^3$ and the conifold $X = V(zw - xy) \subseteq \mathbb{P}^4$. Proposition 3.6 implies that the blow-ups $g : \operatorname{Bl}_{V(x,z)} Q \to Q$ and $f : \operatorname{Bl}_{V(x,z)} X \to X$ are crepant, and hence define crepant resolutions of Q and X, respectively. Indeed, the centers of the blow-ups have codimension 2 in the ambient spaces $\mathbb{P}^3$ and $\mathbb{P}^4$, while the equations defining Q and X have vanish to order exactly 1 at the generic points of the centers $V(x, z)$. On the other hand, Proposition 3.6 implies that blowing up the conifold at its singular point $V(x, y, z, w) = [0; 0; 0; 0; 1]$ does not yield a crepant resolution. Indeed, the equation $zw = y^2$ has vanishes to order exactly 2 at $[0; 0; 0; 0; 1]$, while $[0; 0; 0; 0; 1]$ has codimension $4 \neq 2+1$ in $\mathbb{P}^4$. Intuitively, blowing up the singular point $[0; 0; 0; 0; 1]$ modifies the conifold excessively—by surgically inserting a surface instead of merely a curve at the singular point—and hence changes the canonical class.

To prove Proposition 3.6, we apply standard results on how the canonical classes of nonsingular varieties change on passing to divisors and on blow-ups with nonsingular centers.

Lemma 3.8 (Adjunction formula [49, Proposition 5.73])**.** *Let X' be a nonsingular, quasi-projective*[23] *variety over a field and let X be a prime divisor on X'. If X is normal, then we have that $K_X = c_1(i^*\mathcal{O}_{X'}(K_{X'} + [X]))$, where $i : X \hookrightarrow X'$ is the closed embedding.*

[23]Kollár and Mori [49, Proposition 5.73] have assumed that X' is projective, but the same logic applies for quasi-projective X'.

Lemma 3.9 ([31, Exercise II.8.5]). *If X is a nonsingular variety and $Z \subseteq X$ is nonsingular subvariety, then we have that $K_{\operatorname{Bl}_Z X} = \pi^* K_X + (\operatorname{codim}_Z X - 1)[E]$, where $\pi : \operatorname{Bl}_Z X \to X$ is the projection and E is the exceptional divisor of π.*

Proof of Proposition 3.6. Let $i : X \hookrightarrow X'$ denote the closed embedding. The blow-up $\operatorname{Bl}_Z X$ is the proper transform of X in $\operatorname{Bl}_Z X'$. Hence, $\operatorname{Bl}_Z X = V(\pi^* f / e^m)$ in $\operatorname{Bl}_Z X'$, where $\pi : \operatorname{Bl}_Z X' \to X'$ is the projection, e cuts out the exceptional locus E of π, and m is the multiplicity of f at the generic point of Z. By abuse of notation, we write $i : \operatorname{Bl}_Z X \hookrightarrow \operatorname{Bl}_Z X'$ for the closed embedding and $\pi : \operatorname{Bl}_Z X \to X$ for the projection. We have that

$$\begin{aligned} K_{\operatorname{Bl}_Z X} &= i^*(K_{\operatorname{Bl}_Z X'} + \pi^*[X] - m[E]) \\ &= i^*(\pi^* K_{X'} + (\operatorname{codim}_X Z - 1)[E] + \pi^*[X] - m[E]) \\ &= (\operatorname{codim}_X Z - 1 - m) i^*[E] + \pi^*(i^*(K_{X'} + [X])) \\ &= (\operatorname{codim}_X Z - 1 - m) i^*[E] + \pi^* K_X, \end{aligned}$$

where the first and fourth equalities follow from using Lemma 3.8 to compute the canonical classes of $\operatorname{Bl}_Z X$ and X, respectively, and the second equality follows from Lemma 3.9. The proposition follows. □

We will also need to deal with the canonical classes of resolutions of (the spectra of) Noetherian local rings. As the relevant schemes are not algebraic varieties, we use the dualizing sheaf as a replacement for the canonical class.

Our treatment of the theory of dualizing sheaves is brief and follows the Stacks Project [76, Tag 0DWE]. Let X be a Noetherian scheme. Let $D(X)$ denote the full subcategory of the derived category of $\mathcal{O}_X$-modules consisting of objects with quasi-coherent cohomology sheaves, and denote by $D^+(X)$ the full subcategory of $D(X)$ consisting of objects that are bounded below. Let $f : Y \to X$ be a quasi-projective morphism between Noetherian schemes, and suppose that f factors as $f = \overline{f} \circ i$ where $i : Y \to \overline{Y}$ is an open immersion and $\overline{f} : \overline{Y} \to X$ is projective. Let $R\overline{f}_* : D(\overline{Y}) \to D(X)$ denote the derived pushforward functor, and let $\overline{f}^! : D^+(X) \to D^+(\overline{Y})$ denote the restriction of the right adjoint of $R\overline{f}_*$ to $D^+(X)$—which exists by [76, Tag 0A9E]. We define a functor $f^! : D^+(X) \to D^+(Y)$ by $f^!(-) = \overline{f}^!(-)\Big|_Y$—a functor that is independent of $\overline{Y}$ up to canonical natural isomorphism by [76, Tag 0AA0]. The *relative dualizing complex* is $\omega^\bullet_{Y/X} = f^! \mathcal{O}_X$. If $\omega^\bullet_{Y/X} \simeq \omega_{Y/X}[n]$ for some coherent sheaf

$\omega_{Y/X}$ and integer n, then we call $\omega_{Y/X}$ the *dualizing sheaf of* Y/X.

The canonical class and the dualizing sheaf are well-behaved and essentially coincide for normal, Gorenstein varieties. Recall that a Noetherian local ring R is *Gorenstein* if $\operatorname{Ext}^i(k, R) = 0$ for some $i > \dim R$, where k is the residue field of R and $\dim R$ is the Krull dimension of R. A Noetherian scheme X is *Gorenstein* if $\mathcal{O}_{X,p}$ is Gorenstein for all $p \in X$. Here, $\mathcal{O}_{X,p}$ denotes the local ring of X at p.[24] For normal, Gorenstein, quasi-projective varieties, the canonical class is Cartier and the dualizing sheaf is the corresponding invertible sheaf.

Lemma 3.10 ([49, Corollary 5.69 and Proposition 5.75] and [76, Tag 0C08])**.**

(a) If X is a normal, Gorenstein variety, then K_X is Cartier.[25]

(b) If furthermore X is quasi-projective,[26] *then $X/\operatorname{Spec}\mathbb{C}$ has a dualizing sheaf and $\omega_{X/\operatorname{Spec}\mathbb{C}}$ is the invertible sheaf corresponding to K_X.*

When X/S has an invertible dualizing sheaf, there is a simple relationship between the dualizing sheaves of Y/X and of Y/S: the first Chern class of the dualizing sheaf of Y/X is the difference between the first Chern classes of the dualizing sheaves of Y/S and X/S.

Proposition 3.11. *Let $f : Y \to X$ and $g : X \to S$ be quasi-projective morphisms between Noetherian schemes, and suppose that X/S has an invertible dualizing sheaf. Then, Y/X has a dualizing sheaf if and only if Y/S has a dualizing sheaf. Under the equivalent conditions of the previous sentence, we have that $\omega_{Y/S} \simeq \omega_{Y/X} \otimes \omega_{X/S}$.*

Proof. Let n be such that $\omega^\bullet_{X/S} \simeq \omega_{X/S}[n]$. As $\omega_{X/S}$ is invertible, the complex $\omega_{X/S}[n]$ is perfect. [76, Tag 0ATX] implies that $\omega^\bullet_{Y/S} \simeq f^!\omega^\bullet_{X/S}$. Hence, we have that

$$f^!\omega^\bullet_{X/S} \simeq f^!(\mathcal{O}_X \otimes^{\mathbf{L}}_{\mathcal{O}_X} \omega^\bullet_{X/S}) \simeq (f^!\mathcal{O}_X) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Lf^*\omega^\bullet_{X/S},$$

where $- \otimes^{\mathbf{L}} -$ denotes the derived tensor product, $Lf^*(-)$ denotes the derived pullback, and the second isomorphism is by [76, Tag 0A9T]. Because $\omega_{X/S}$ is invertible, we have that $Lf^*\omega^\bullet_{X/S} \simeq (f^*\omega_{X/S})[n]$. Therefore, we have that $\omega^\bullet_{Y/S} \simeq \omega^\bullet_{Y/X} \otimes \omega_{X/S}$, and the lemma follows. □

[24] As the localization of a Gorenstein local ring is Gorenstein [14, Corollary 21.17], the property of being Gorenstein can be checked at closed points.

[25] Lemma 3.10(a) implies in particular that normal, Gorenstein varieties are $\mathbb{Q}$-Gorenstein.

[26] Kollár and Mori [49, Proposition 5.75] have assumed that X is projective, but the same logic applies for quasi-projective X.

In light of Lemma 3.10, it follows from Proposition 3.11 that the dualizing sheaf of a crepant morphism between normal, Gorenstein, quasi-projective varieties is an invertible sheaf that defines a torsion class in the Picard group.

Proposition 3.12. *Let $f : Y \to X$ be a morphism between normal, Gorenstein, quasi-projective varieties. If f is crepant, then Y/X has a invertible dualizing sheaf $\omega_{Y/X}$ whose class in* $\operatorname{Pic} Y$ *is torsion.*

Proof. Lemma 3.10(a) implies that K_X and K_Y are Cartier. As f is crepant, we have that $f^*K_X = K_Y$ in $\mathrm{Cl}(Y) \otimes \mathbb{Q}$. Hence, the class of $\mathcal{O}_Y(K_Y) \otimes f^*\mathcal{O}_X(K_X)^*$ in $\operatorname{Pic} Y$ must be torsion, where $\mathcal{O}_X(K_X)$ and $\mathcal{O}_Y(K_Y)$ are the invertible sheaves corresponding to K_X and K_Y, respectively. The proposition follows by Proposition 3.11. □

Similarly to how the canonical class of an open subvariety is the restriction of the canonical class of the ambient variety, dualizing sheaves restrict nicely under localization. Specifically, the dualizing sheaf of the base-change to the spectrum of a stalk is the restriction/localization of the dualizing sheaf of the original scheme.

Proposition 3.13. *Let X be a quasi-compact, quasi-separated, Noetherian scheme, let $f : Y \to X$ be a quasi-projective morphism, and let $p \in X$ be a point. If $\omega_{Y/X}$ is a dualizing sheaf for Y/X, then $j^*\omega_{Y/X}$ is a dualizing sheaf for $(Y \times_X \operatorname{Spec} \mathcal{O}_{X,p})/\operatorname{Spec} \mathcal{O}_{X,p}$, where $j : Y \times_X \operatorname{Spec} \mathcal{O}_{X,p} \to Y$ is the natural morphism.*

Proof. Follows from [76, Tags 0A9N and 0A9P]. □

3.4 Du Val singularities

For this subsection, we consider resolutions of general two-dimensional local rings with Du Val singularities following Lipman [54]—who extended the results of Du Val [13] and Artin [2] on the resolutions of Du Val singularities to local rings without algebraically closed residue fields. As we will see in Sections 3.5 and 4.1, such local rings arise from threefolds with cDV singularities as the local rings at codimension 2 points of the singular locus.

We say that a Noetherian, normal, local ring R has *rational singularities* if there is a resolution $f : X \to \operatorname{Spec} R$ such that $H^1(X, \mathcal{O}_X) = 0$. Likewise, we say that a Noetherian scheme X has *rational singularities* if $\mathcal{O}_{X,p}$ has rational singularities for all $p \in X$.[27] By the *dimension* of a Noetherian local ring, we mean the Krull dimension. We say that a two-dimensional Noetherian local ring R has a *Du Val singularity* if R is Gorenstein and has rational singularities.

Example 3.14 (The quadric cone has a Du Val singularity). As in Example 3.1, consider the quadric cone $Q = V(z^2 - xy) \subseteq \mathbb{P}^3$. The local ring $\mathcal{O}_{Q,[0;0;0;1]}$ has a Du Val singularity. To see this, note that Q is a hypersurface and is therefore Gorenstein. Letting $X = \mathrm{Bl}_{V(x,z)} \operatorname{Spec} \mathcal{O}_{Q,[0;0;0;1]}$, one can show that $H^1(X, \mathcal{O}_X) = 0$, so that $\mathcal{O}_{Q,[0;0;0;1]}$ has rational singularities.

The first result shows that there is a minimal resolution for every two-dimensional, normal, Noetherian local ring with rational singularities.

Theorem 3.15 ([54, Theorem 4.1])**.** *Let R be a two-dimensional Noetherian local ring with rational singularities. There exists a resolution $f : Y \to \operatorname{Spec} R$ such that every resolution $f' : Y' \to \operatorname{Spec} R$ factors through f.*

We call the resolution given in Theorem 3.15 the *minimal resolution* of $\operatorname{Spec} R$.

There is a simple characterization of the minimal resolution of a Du Val singularity in terms of dualizing sheaves. Loosely speaking, the minimal resolution is the unique crepant resolution. The formal result uses dualizing sheaes to generalize the fact that the minimal resolutions are the unique crepant resolutions of complex algebraic surfaces with Du Val singularities to the setting of local rings.

Proposition 3.16. *Let R be a local ring with a Du Val singularity that is of essentially finite type over a field.*[28]

(a) If $g : X \to \operatorname{Spec} R$ is the minimal resolution, then $X/\operatorname{Spec} R$ has a dualizing sheaf $\omega_{X/\operatorname{Spec} R} \simeq \mathcal{O}_X$.

(b) If $f : Y \to \operatorname{Spec} R$ is a non-minimal resolution, then $Y/\operatorname{Spec} R$ has an invertible dualizing sheaf and the class of $\omega_{Y/\operatorname{Spec} R}$ in $\operatorname{Pic} Y$ is non-torsion.

[27] In fact, the property of having rational singularities can be checked at closed points. To justify this assertion, it suffices to show that the localizations of a Noetherian local rings with rational singularities have rational singularities. Consider a Noetherian local ring R with rational singularities and let $f : X \to \operatorname{Spec} R$ be a resolution with $H^1(X, \mathcal{O}_X) = 0$. Let $\mathfrak{p}$ be a prime ideal in R. The base-change $f \times_{\operatorname{Spec} R} \operatorname{Spec} R_{\mathfrak{p}}$ is a resolution of singularities, and we have that $H^1(X \times_{\operatorname{Spec} R} \operatorname{Spec} R_{\mathfrak{p}}, \mathcal{O}_{X \times_{\operatorname{Spec} R} \operatorname{Spec} R_{\mathfrak{p}}}) = 0$ by the flat base change [76, Tag 02KH]. It follows that $R_{\mathfrak{p}}$ has rational singularities, as desired.

[28] The same argument applies if R is of essentially finite type over $\mathbb{Z}$.

Although Proposition 3.16 is standard, we provide a proof of Proposition 3.16 in Appendix A for sake of completeness.

Example 3.17 (Minimal resolution of the local ring of the quadric cone at its singular point). As in Example 3.1, consider the quadric cone $Q = V(z^2 - xy) \subseteq \mathbb{P}^3$. Let $f : \mathrm{Bl}_{V(x,z)} Q \to Q$ denote the blow-up of Q along $V(x, z)$. Consider the resolution $f \times_Q \operatorname{Spec} \mathcal{O}_{Q,[0;0;0;1]} : \mathrm{Bl}_{V(x,z)} \operatorname{Spec} \mathcal{O}_{Q,[0;0;0;1]} \to \operatorname{Spec} \mathcal{O}_{Q,[0;0;0;1]}$ of $\operatorname{Spec} \mathcal{O}_{Q,[0;0;0;1]}$. We claim that $f \times_Q \operatorname{Spec} \mathcal{O}_{Q,[0;0;0;1]}$ is the minimal resolution of $\operatorname{Spec} \mathcal{O}_{Q,[0;0;0;1]}$. To see this, note that f is crepant by Proposition 3.6 (as we showed in Example 3.7). By Propositions 3.12 and 3.13, it follows that $\mathrm{Bl}_{V(x,z)} \operatorname{Spec} \mathcal{O}_{Q,[0;0;0;1]} / \operatorname{Spec} \mathcal{O}_{Q,[0;0;0;1]}$ has an invertible dualizing sheaf whose class in $\operatorname{Pic} \mathrm{Bl}_{V(x,z)} \operatorname{Spec} \mathcal{O}_{Q,[0;0;0;1]}$ is torsion. Hence, the contrapositive of Proposition 3.16(b) guarantees that $f \times_Q \operatorname{Spec} \mathcal{O}_{Q,[0;0;0;1]}$ is the minimal resolution of $\operatorname{Spec} \mathcal{O}_{Q,[0;0;0;1]}$.

Lipman [54] has determined the possible intersection matrices of the exceptional curves of minimal resolutions of local rings with Du Val singularities. To fix notation, let R be a two-dimensional Noetherian local ring with rational singularities, and let $f : X \to \operatorname{Spec} R$ be the minimal resolution of $\operatorname{Spec} R$. Let $E_1, E_2, \dots, E_k$ be the exceptional curves of f. Consider the free abelian group $\mathfrak{L}(R)$ with basis $\Delta(R) = \{\mathbf{r}_1, \dots, \mathbf{r}_k\}$. Define a homomorphism $\theta_R : \mathfrak{L}(R) \to \operatorname{Pic} X$ by $\theta_R(\mathbf{r}_i) = c_1^{-1}([E_i])$. It turns out that θ_R is injective and has finite cokernel.

Proposition 3.18 ([54, Lemma 14.1 and Proposition 17.1])**.** *If R is a two-dimensional Noetherian local ring with rational singularities, then the homomorphism θ_R is injective and has finite cokernel.*

In light of Proposition 3.18, the tensor product $\theta_{R,\mathbb{Q}} = \theta_R \otimes \mathbb{Q} : \mathfrak{L}(R) \otimes \mathbb{Q} \to \operatorname{Pic} X \otimes \mathbb{Q}$ has an inverse, which we denote by $\theta_{R,\mathbb{Q}}^{-1}$. Continuing with the same notation, given $1 \le i \le j \le k$, consider the intersection numbers

$$E_i \cdot E_j = \deg_{E_i} \mathcal{L}_j = \begin{array}{l} \text{degree of the invertible sheaf } \mathcal{L}_j \text{ restricted} \\ \text{to the proper one-dimensional scheme } E_i \end{array},$$

where $\mathcal{L}_j = c_1^{-1}([E_j]) \in \operatorname{Pic} X$. Define a $k \times k$ matrix $M = M(R) = (m_{i,j})$ by $m_{i,j} = E_i \cdot E_j$. The matrix $M(R)$ is symmetric as $m_{i,j}$ counts the number of intersections between E_i and E_j for $i \neq j$.[29] We equip $\mathfrak{L}(R)$ and $\mathfrak{L}(R) \otimes \mathbb{R}$ with the bilinear form given whose matrix expansion in the basis $\Delta(R) \subseteq \mathfrak{L}(R)$ is $-M(R)$.

[29]To be precise, let $1 \le i \neq j \le k$ be indices. As X is nonsingular and E_i and E_j meet dimension-

Du Val [13] showed that $M(R)$ is negative definite (see also [54, Lemma 14.1]). When R is Gorenstein, $-M(R)$ is even the Killing matrix of a root system.

Theorem 3.19 ([54, §24])**.** *Let R be a local ring with a Du Val singularity.*[30]

(a) $-M(R)$ is the Killing matrix of an irreducible root system (up to simultaneous permutation of the rows and columns).

(b) If the residue field of R is algebraically closed and of characteristic 0, then $-M(R)$ is the Killing matrix of a simply-laced irreducible root system (up to simultaneous permutation of the rows and columns).[31]

In light of Theorem 3.19, we obtain a canonical root system in $\mathfrak{L}(R)$ for every local ring R with Du Val singularities.

Corollary 3.20. *Let R be a local ring with a Du Val singularity. There exists a unique set $\mathfrak{R} \subseteq \mathfrak{L}(R)$ that forms a root system in $\mathfrak{L}(R) \otimes \mathbb{R}$ of which $\Delta(R)$ is a set of simple roots.*

Proof. Follows from Corollary 2.3 and Theorem 3.19. □

Remark 3.21. Corollary 3.20 implies that the dual graph of the exceptional locus of the minimal resolution of a Du Val singularity can be interpreted as a Dynkin diagram. Indeed, Corollary 3.20 guarantees that the exceptional curves naturally correspond to the simple roots of an irreducible root system. Simple roots in turn correspond to vertices of a Dynkin diagram.

We denote the root system $\mathfrak{R}$ of Corollary 3.20 by $\mathfrak{R}(R)$. Theorem 3.19(b) implies that $\mathfrak{R}(R)$ is simply-laced whenever R is a local ring with a Du Val singularity and an algebraically closed residue field.

Example 3.22 (The quadric cone as an A_1 singularity). As in Example 3.1, consider the quadric cone $Q = V(z^2 - xy) \subseteq \mathbb{P}^3$. We showed in Example 3.17 that the minimal resolution of $\operatorname{Spec} \mathcal{O}_{Q,[0;0;0;1]}$ contains one exceptional curve. Hence, $-M(\mathcal{O}_{Q,[0;0;0;1]})$ must be the Killing matrix of type A_1, because the unique root system in $\mathbb{R}^1$ is of type A_1. Du Val singularities of type A_1 are also called *ordinary double points.*

ally transversely, $m_{i,j}$ is the length of $H^0(X, \mathcal{O}_{E_i \cap E_j})$ as an R-module due to the characterization of the intersection number by Fulton [26, Proposition 8.2(b)].

[30] Strictly speaking, Lipman [54] has classified the possible exceptional loci of minimal resolutions of surfaces with rational double points. In dimension 2, rational double points are precisely the rational Gorenstein singularities [58, Corollary 4-6-16].

[31] Theorem 3.19(b) is originally due to Du Val [13].

We will also use a necessary and sufficient condition for invertible sheaves on resolutions to be globally generated. Intuitively, an invertible sheaf is relatively globally generated if and only if it is relatively nef.

Proposition 3.23 ([54, Proposition 1.2 and Theorem 12.1])**.** *Let R be a two-dimensional Noetherian local ring with rational singularities, let $\pi : X \to \operatorname{Spec} R$ be a resolution, and let $\mathcal{L} \in \operatorname{Pic} X$ be an invertible sheaf. The natural homomorphism $\pi^*\pi_*\mathcal{L} \to \mathcal{L}$ is surjective if and only if $\deg_C \mathcal{L} \geq 0$ holds for all exceptional curves C.*

3.5 Compound Du Val singularities

Our treatment of compound Du Val (cDV) singularities focuses on threefolds and follows Reid [70, 71]. Intuitively, a cDV singularity is a singularity whose general hyperplane section is a Du Val singularity.

Let W be a normal threefold. Intuitively, a hyperplane section through p is a subvariety that is cut out in a neighborhood of p by a single equation whose differential does not vanish at p. Formally, a *hyperplane section through a closed point* $p \in W$ is a prime Cartier divisor H containing p such that $\dim \mathfrak{m}_{H,p}/\mathfrak{m}_{H,p}^2 = \dim \mathfrak{m}_{W,p}/\mathfrak{m}_{W,p}^2 - 1$. Here, $\mathfrak{m}_{H,p}$ (resp. $\mathfrak{m}_{W,p}$) denotes the maximal ideal of $\mathcal{O}_{H,p}$ (resp. $\mathcal{O}_{W,p}$). We say that W has *cDV singularities* at a closed point $p \in W$ if there exists a hyperplane section H through p such that $\mathcal{O}_{H,p}$ has a Du Val singularity. As the class of Du Val singularities is closed under analytic local isomorphisms, it is equivalent to require that $\mathcal{O}_{H,p}$ has a Du Val singularity for the general hyperplane section through p.

Proposition 3.24 ([71, §0.5])**.** *If a threefold W has cDV singularities at a closed point p, then $\mathcal{O}_{H,p}$ has a Du Val singularity for all hyperplane sections H through p whose tangent planes are general (in the sense of Reid [70, Definition 2.5]).*

Proof sketch. Reid [70, §2] has shown that W is analytically locally isomorphic (at p) to the variety in $\mathbb{C}^4$ cut out by $f + wg$, where $f \in \mathbb{C}[x, y, z]$ is the equation of a Du Val singularity and $g \in \mathbb{C}[x, y, z, w]$ is arbitrary. It follows that the general hyperplane section H through p is analytically locally isomorphic (at p) to a Du Val singularity. But Du Val singularities are the only singularities that are analytically locally isomorphic to Du Val singularities. □

A basic example of a cDV singularity is the singularity of the conifold.

Example 3.25 (The conifold has a cDV singularity). As in Example 3.1, consider the conifold $X = V(zw - xy) \subseteq \mathbb{P}^4$. The prime divisor $H = V(z - w) \cap X$ is a hyperplane section through p. Note that H is cut out in $V(z - w) \subseteq \mathbb{P}^4$ by the equation $z^2 - xy$. Hence, the hyperplane section H is isomorphic to the quadric cone, which has a Du Val singularity (as we showed in Example 3.14. Because the quadric cone has a singularity of type A_1 (as we showed in Example 3.22), we say that the conifold has a *compound* A_1 *singularity.*

If W has *cDV singularities* at all closed points, then we say that W has *cDV singularities.* Threefolds with cDV singularities have rational, Gorenstein singularities.

Proposition 3.26 ([70, Theorem 2.6(II)])**.** *If W is a threefold with cDV singularities, then W is Gorenstein and has rational singularities.*

Hence, threefolds with cDV singularities have well-behaved canonical classes. Moreover, the local rings of such threefolds at codimension 2 points have Du Val singularities—due to the definitions of rational singularities and Gorenstein schemes and because Du Val singularities are the rational Gorenstein surface singularities.

Reid [71] has shown that crepant partial resolutions of varieties with cDV singularities give rise (locally) to crepant partial resolutions of hyperplane sections that have Du Val singularities.

Theorem 3.27 ([71, Theorem 1.14])**.** *Let $f : Y \to W$ be a crepant partial resolution of a threefold W with cDV singularities. Let $p \in W$ be a closed point and let H be a hyperplane section through p. If $\mathcal{O}_{H,p}$ has a Du Val singularity, then $Y \times_W \operatorname{Spec} \mathcal{O}_{H,p}$ is integral and normal and the minimal resolution of* $\operatorname{Spec} \mathcal{O}_{H,p}$ *factors through*

$$f \times_W \operatorname{Spec} \mathcal{O}_{H,p} : Y \times_W \operatorname{Spec} \mathcal{O}_{H,p} \to \operatorname{Spec} \mathcal{O}_{H,p}.$$

Example 3.28 (A crepant resolution of the conifold resolves a Du Val hyperplane section). As in Example 3.1, consider the conifold $X = V(zw - xy) \subseteq \mathbb{P}^4$. We showed in Example 3.7 that $f : \operatorname{Bl}_{V(x,z)} X \to X$ is a crepant resolution of X. Let $H = V(z - w) \cap X$, which we showed in Example 3.25 to have a Du Val singularity of type A_1 at $V(x, y, z, w) = [0; 0; 0; 0; 1]$. The base change $\operatorname{Bl}_{V(x,z)} X \times_X \operatorname{Spec} \mathcal{O}_{H,[0;0;0;0;1]}$ is isomorphic the blow-up $\operatorname{Bl}_{V(x,z)} \operatorname{Spec} \mathcal{O}_{H,[0;0;0;0;1]}$, which is normal. Moreover, the base change $f \times_X \operatorname{Spec} \mathcal{O}_{H,p}$ is isomorphic to the projection $\operatorname{Bl}_{V(x,z)} \operatorname{Spec} \mathcal{O}_{H,[0;0;0;0;1]} \to \operatorname{Spec} \mathcal{O}_{H,[0;0;0;0;1]}$, which we showed in Example 3.17 to be minimal resolution of an

A_1 singularity. In general, the base change of a crepant resolution to a hyperplane section with a Du Val singularity may only be a partial resolution through which the minimal resolution factors—as guaranteed by Theorem 3.27—instead of the full minimal resolution—as in this example.

It follows from Theorem 3.27 that every crepant partial resolution of a threefold with cDV singularities continues to have cDV singularities.

Corollary 3.29. *If $f : Y \to W$ is a crepant partial resolution of a threefold W with cDV singularities, then Y has cDV singularities.*

To prove Corollary 3.29, we use the observation of Reid [72, §1] that any partial resolution of the minimal resolution of a Du Val singularity must itself have Du Val singularities at every closed point.

Proof. Let $p \in Y$ be an arbitrary closed point. We need to show that there exists a hyperplane section $H \subseteq Y$ through p such that $\mathcal{O}_{H,p}$ has a Du Val singularity.

Let $p' = f(p)$. Because W has cDV singularities, there is a hyperplane section $H' \subseteq W$ through p' such that $\mathcal{O}_{H',p'}$ has a Du Val singularity. Theorem 3.27 guarantees that $Y \times_W \operatorname{Spec} \mathcal{O}_{H',p'}$ is integral and normal and that the minimal resolution of $\operatorname{Spec} \mathcal{O}_{H',p'}$ factors through $f \times_W \operatorname{Spec} \mathcal{O}_{H',p'}$. [72, §1] implies that $\mathcal{O}_{Y \times_W \operatorname{Spec} \mathcal{O}_{H',p'},p}$ has a Du Val singularity. Taking $H = f^{-1}(H')$, we have that $\mathcal{O}_{Y \times_W \operatorname{Spec} \mathcal{O}_{H',p'},p} = \mathcal{O}_{H,p}$ as quotients of $\mathcal{O}_{Y,p}$. Hence, the local ring $\mathcal{O}_{H,p}$ must have a Du Val singularity as well. As p was arbitrary, we have shown that Y must have cDV singularities. □

3.6 Terminal singularities

Our treatment of terminal singularities follows Matsuki [58]. Intuitively, terminal singularities are the singularities such that any exceptional divisor of any proper birational morphism from a nonsingular variety appears with positive multiplicity in the canonical class.

Formally, let X be a $\mathbb{Q}$-Gorenstein variety and let $f : Y \to X$ be a proper birational morphism from a nonsingular variety Y. Let $E_1, \ldots, E_k$ be the exceptional divisors of f. Supposing that mK_X is Cartier, we write

$$mK_Y = f^*(mK_X) + m \sum_{i=1}^{k} a(E_i, X, Y)[E_i]$$

in $\operatorname{Cl} Y$, where $a(E_i, X, Y) \in \frac{1}{m}\mathbb{Z}$. It turns out that the quantity $a(E_i, X, Y)$ is independent of Y (see, e.g., [58, Proposition-Definition 4-4-1]), and hence we can write $a(E_i, X, Y)$ as $a(E_i, X)$. We call $a(E, X)$ the *discrepancy of X at E* and say that X has *terminal singularities* if $a(E, X) > 0$ for all exceptional divisors E of all proper birational morphisms $f : Y \to X$ from nonsingular varieties Y.

Recall that nonsingular varieties have terminal singularities. In dimension 2, the converse is also true—there are no nontrivial terminal singularities.

Proposition 3.30 ([58, Corollary 4-6-6])**.** *Varieties with terminal singularities are nonsingular in codimension 2.*

In dimension 3, on the other hand, there are nontrivial terminal singularities. In particular, any isolated cDV singularity is terminal.

Proposition 3.31 ([71, Theorem 1.1])**.** *If X is a threefold with isolated cDV singularities, then X has terminal singularities.*

For example, Proposition 3.31 implies that the conifold has terminal singularities.

Remark 3.32. A partial converse to Proposition 3.31 holds: every Gorenstein threefold with terminal singularities has isolated cDV singularities [49, Corollary 5.38].

Consider a crepant partial resolution of a threefold with cDV singularities. In light of Corollary 3.29 and Proposition 3.31, if the partial resolution has isolated singularities, then it must have terminal singularities.

Corollary 3.33. *Let $f : Y \to W$ is a crepant partial resolution of a threefold with cDV singularities. If Y has isolated singularities, then Y has terminal singularities.*

Proof. Follows from Corollary 3.29 and Proposition 3.31. □

The class of singularities that arise in the minimal model program are the terminal singularities of $\mathbb{Q}$-factorial varieties—i.e., the *$\mathbb{Q}$-factorial terminal singularities.* Such singularities cannot be resolved without moving the canonical class further from being nef (see Kovács [50]), which is the motivation for considering crepant partial resolutions with $\mathbb{Q}$-factorial terminal singularities. 3.33 will be useful in building crepant partial resolutions with $\mathbb{Q}$-factorial terminal singularities in our examples (see Section 8).

3.7 Flops

Our treatments of flops and (relative) minimal models follow Kollár and Mori [49].

Let Y be a $\mathbb{Q}$-Gorenstein variety. We say that a proper birational morphism $f : Y \to X$ is a *flopping contraction* if K_Y is numerically f-trivial and the exceptional locus of f has codimension at least 2 (in Y). If furthermore D is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor of Y such that $-D$ is f-ample, then we say that f is a *D-flopping contraction.* In this case, we say that a birational morphism $f^+ : Y^+ \to X$ is a *D-flop of f* if Y^+ is $\mathbb{Q}$-Gorenstein, D is $\mathbb{Q}$-Cartier on Y^+ and f^+-ample, and the exceptional locus of f^+ has codimension at least 2 (in Y^+). Here, we are using the fact that the composite $f^{-1} \circ f^+ : Y^+ \dashrightarrow Y$—which is a rational map[32]—is an isomorphism in codimension 1 to identify D with a $\mathbb{Q}$-divisor on Y^+. Being an isomorphism in codimension 1, the composite $f^{-1} \circ f^+$ is a codimension 2 surgery operation—which is the sense in which flops are surgery operations.

The simplest example of a flop is due to Atiyah [3].

Example 3.34 (Crepant resolutions of the conifold and Atiyah's flop)*.* As in Example 3.1, consider the conifold $X = V(zw - xy) \subseteq \mathbb{P}^4$. Let $Y = \mathrm{Bl}_{V(x,z)} X$ denote the blow-up of X along the line $V(x, z)$, and let $f : Y \to X$ denote the projection. The exceptional locus of f is a curve and—as we showed in Example 3.7—f is crepant. Hence, f is a flopping contraction. Let $D = f^{-1}(V(x, z))$ denote the prime divisor above the center of the blow-up, which is Cartier. As we showed in Example 3.5, the divisor $-D$ is f-ample. Hence, f is a D-flopping contraction.

We now construct the D-flop of f. Let $Y^+ = \mathrm{Bl}_{V(x,w)} X$ denote the blow-up of X along the line $V(x, w)$, and let $f^+ : Y^+ \to X$ denote the projection. We claim that f^+ is the D-flop of f. By symmetry between z and w, the exceptional locus of f^+ is a curve and K_{Y^+} is numerically f^+ trivial. Hence, to prove that f^+ is the D-flop of f, it suffices to show that D is f^+-ample. By symmetry, we can instead show that $D' = f^{-1}(V(x, w))$ is f-ample. Note that $D + D'$ is linearly equivalent to $f^{-1}(V(x))$, which is numerically f-trivial. Hence, D' is numerically equivalent to $-D$ over X, which implies that D' is f-ample by the relative Kleiman criterion (Theorem 3.3).

The flop that we described is called *Atiyah's flop.* It relates the two crepant resolutions of the conifold.

The flops of extremal flopping contractions are especially well-behaved. Recall

[32]We denote morphisms by solid arrows $\to$ and rational maps by dashed arrows $\dashrightarrow$.

that a birational morphism $f : Y \to X$ is *extremal* if Y is $\mathbb{Q}$-factorial and f has relative Picard number $\dim N^1(Y/X) = 1$. Intuitively, extremal flopping contractions are flopping contractions that contract as little of the variety as possible.

Theorem 3.35 (Uniqueness of flops [49, Corollary 6.4]). *Let $f : Y \to X$ be a D-flopping contraction.*

(a) If a D-flop of f exists, then it is unique.

(b) If f is extremal, then the D-flop of f does not depend on D.

If f is an extremal flopping contraction, then we call the D-flop of f—for any $\mathbb{Q}$-Cartier divisor D such that f is a D-flopping contraction—the *flop* of f. In dimension 3, terminal flops exist—as shown by Kawamata [39] and Mori [62]—and preserve nonsingularity—as shown by Kollár [48].

Theorem 3.36 (Terminal flops between threefolds [49, Theorems 6.14 and 6.15]). *Let Y be a threefold and let $f : Y \to X$ be a D-flopping contraction.*

(a) If Y has terminal singularities, then f has a D-flop $f^+ : Y^+ \to X$.[33]

(b) If Y is nonsingular above a closed point $p \in X$, then the variety Y^+ defined in Part (a) is also nonsingular above p.[34]

If $\pi : Y \to W$ and $\pi^+ : Y^+ \to W$ are morphisms, we say that π and π^+ are *related by an extremal flop* if there exist a morphism $p : X \to W$ and factorizations $\pi = p \circ f$ and $\pi^+ = p \circ f^+$ such that f is an extremal flopping contraction with flop f^+. We say that $\pi : Y \to W$ and $\pi' : Y' \to W$ are *related by a finite sequence of extremal flops* if there exist morphisms $\pi_i : Y_i \to W$ for $i = 1, 2, \ldots, n-1$ such that π_j and π_{j+1} are related by an extremal flop for all $0 \leq j \leq n-1$, where $\pi_0 = \pi$ and $\pi_n = \pi'$.

[33] It follows from the main result of Shokurov [75] that Theorem 3.36(a) holds in dimension 4. More generally, Birkar et al. [5] and Hacon and McKernan [30] have shown existence of terminal flips–and hence the existence of terminal flops—in all dimensions.

[34] Kollár [48] actually showed, more generally, that flops between threefolds preserve the analytic singularity type. However, the proof of Theorem 3.36(b) given by Kollár [48] relies on an analytic classification of terminal flops in dimension 3, and hence only applies to threefolds.

3.8 Minimal models

Following Mori [61], we say that a projective morphism $\pi : Y \to W$ is a *(relative) minimal model* if Y is $\mathbb{Q}$-factorial and has terminal singularities and K_Y is π-nef. Given a morphism $\pi : Y \to W$, a *minimal model of* π consists of a minimal model $\pi' : Y' \to W$ and a birational map $h : Y \dashrightarrow Y'$ such that $\pi' \circ h = \pi$ as rational maps.

Any two birational relative minimal models are isomorphic in codimension 1 [58, Proposition 12-1-2]. Moreover, if $\pi : Y \to W$ is a minimal model, and π' is a minimal model of π, then the birational map $h : Y \dashrightarrow Y'$ induces an isomorphism between the spaces $N^1(Y/W) \cong N^1(Y'/W)$ by taking proper transforms [58, Lemma 12-2-1]. In this case, we can therefore identify $N^1(Y/W)$ and $N^1(Y'/W)$. As minimal models are isomorphic in codimension 1 and the definition of movability depends only on codimension 1 geometry [39, Lemma 2.3], we have that $\overline{\mathrm{Mov}}(Y/W) = \overline{\mathrm{Mov}}(Y'/W)$ under the identification between $N^1(Y/W)$ and $N^1(Y'/W)$ [58, Definition 12-2-5]. The KKMR decomposition—which was proven by Kawamata [39], Kollár [48], Mori [61], and Reid [71]—relates the movable cone $\overline{\mathrm{Mov}}(Y/W)$ to the nef cones of the minimal models of π.

Theorem 3.37 (KKMR Decomposition [58, Theorem 12-2-7])**.** *Let $\pi : Y \to W$ be a minimal model. Let $\{\pi_i : Y_i \to W \mid i \in I\}$ be a set of representatives for the set of isomorphism classes of minimal models of π. The following conclusions hold.*

(a) The cones $\overline{\mathrm{Amp}}(Y_i/W)$ are locally polyhedral in $N^1(Y/W)$.

(b) The cones $\mathrm{Amp}(Y_i/W)$ are pairwise disjoint in $N^1(Y/W)$.

(c) There is a decomposition

$$\overline{\mathrm{Mov}}(Y/W) = \textit{closure of } \bigcup_{i \in I} \overline{\mathrm{Amp}}(Y_i/W) \textit{ in } N^1(Y/W).$$

Roughly, the KKMR decomposition says that (1) any divisor that is nef on one minimal model is ample on any minimal model, (2) no divisor is ample on more than one isomorphic class of minimal models, and (3) a dense set of movable divisors are nef on at least one minimal model. There is also a geometric interpretation of when the nef cones of two minimal models share a codimension 1 face: this occurs precisely when the two minimal models are related by an extremal flop [58, Proposition 12-2-

2]. Hence, the KKMR decomposition reflects the set of relative minimal models and some of the geometric relationships between the (relative) minimal models.

Example 3.38 (The KKMR decomposition for crepant resolutions of the conifold). As in Example 3.1, consider the conifold $X = V(zw - xy) \subseteq \mathbb{P}^4$. Let $Y = \mathrm{Bl}_{V(x,z)} X$ denote the blow-up of X along the line $V(x, z)$, and let $f : Y \to X$ denote the projection. Let $D = f^{-1}(V(x, z))$ denote the prime divisor above the center of the blow-up, which is Cartier. As we showed in Example 3.5, we have that $\mathrm{Mov}(Y/X) = \mathbb{R}[D]$ and that $\overline{\mathrm{Amp}}(Y/X) = \mathbb{R}_{\leq 0}[D]$. As we showed in Example 3.34, we have that $\overline{\mathrm{Amp}}(Y^+/X) = \mathbb{R}_{\geq 0}[D]$, where $Y^+ = \mathrm{Bl}_{V(x,w)} X$ and $f^+ : Y^+ \to X$ is the projection. Theorem 3.37 implies that there are no other minimal models. The two nef cones meet along the codimension 1 face $\{0\} \subseteq N^1(Y/X)$, and the corresponding extremal flop is Atiyah's flop (as defined in Example 3.34).

Remark 3.39. The KKMR decomposition also implies that uniqueness of the relative minimal models of surfaces. When Y is a surface, the closed movable cone $\overline{\mathrm{Mov}}(Y/X)$ coincides with the nef cone $\overline{\mathrm{Amp}}(Y/X)$ (such as in Example 3.4). By Theorem 3.37, there cannot be any other non-isomorphic relative minimal models.

We now explain how flops relate minimal models in dimension 3. Suppose that $\pi : Y \to W$ is a minimal model and that $\pi' : Y' \to W$ is related to π by an extremal flop. As the canonical class is determined in codimension 1 and is numerically trivial along flopping contractions, the canonical class $K_{Y'}$ must be π'-nef. It turns out that Y' must have $\mathbb{Q}$-factorial terminal singularities, so that $\pi' : Y' \to W$ is also a minimal model. Due to the existence of terminal flops, we can attempt to make any movable divisor D nef by repeatedly flopping—yielding a sequence of extremal D-flops. As there is no infinite sequence of extremal D-flops between threefolds with terminal singularities [48], such a sequence must eventually terminate. Hence, in dimension 3, any divisor can be made nef by applying a finite sequence of extremal flops. By the KKMR decomposition, it follows that birational minimal threefolds must be related by finite sequences of extremal flops.

Theorem 3.40 (Flops between minimal threefolds [49, Corollary 6.19])**.** *If Y is a threefold, $\pi : Y \to X$ is a minimal model, and π' is a minimal model of π that is not isomorphic to π, then π and π' are related by a finite sequence of extremal flops.*

As flops preserve nonsingularity in dimension 3, it follows that the nonsingularity of one minimal model of a morphism implies the nonsingularity of all minimal models.

Corollary 3.41 (Singularity types of minimal models [48, Corollary 4.11])**.** *Let* $\pi : Y \to W$ *be a minimal model with* $\dim Y = 3$ *and let* $\pi' : Y' \to W$ *is a minimal model of* π*. If* π *is nonsingular above a closed point* $p \in W$*, then so is* π'*.*

Proof. Follows from Theorems 3.36(b) and 3.40. □

Example 3.42 (Flops and singularity types for relative minimal models of the resolutions of the conifold)*.* As in Example 3.1, consider the conifold $X = V(zw - xy) \subseteq \mathbb{P}^4$. As shown in Example 3.38, there are two relative minimal models of resolutions of the conifold, which are related by Atiyah's flop and are both nonsingular.

A similar but more complex structure persists for the minimal resolutions of *binomial* singularities—that is, singularities of the form $V(zw - x_1x_2x_3)$. In this case, there are multiple minimal models, which are related by generalizations of Atiyah's flop. Esole and Yau [17] have presented examples of crepant resolutions of binomial singularities in F-theory.

4 Construction of "matter representations"

The setup for the remainder of the paper is the following situation.

Situation 4.1. Consider a quasi-projective threefold W with cDV singularities. Let W_{sing} denote the singular locus of W, and let

$$W_{\text{sing}} = W_0 \cup \bigcup_{i=1}^{s} Z_i$$

be the set-theoretic decomposition of W into its irreducible components, where W_0 is a finite set of closed points and $Z_1, \ldots, Z_s$ are curves.[35] Let η_i denote the generic point of Z_i for $1 \leq i \leq s$.

We consider a particularly well-behaved class of partial resolutions, which are the projective crepant partial resolutions with $\mathbb{Q}$-factorial terminal singularities.

Definition 4.2. Given a $\mathbb{Q}$-Gorenstein variety W, a *good* partial resolution of W is a crepant partial resolution $\pi : Y \to W$ such that π is projective and Y is $\mathbb{Q}$-factorial and has terminal singularities.

[35]As W has cDV singularities, it is normal and hence in particular non-singular in codimension 1.

Remark 4.3. Our notion of a good partial resolution is unrelated to the notion of a "good minimal model" from Nakayama [69].

Let Good(W) denote the set of isomorphism classes of good partial resolutions. In this thesis, we investigate what can be deduced about the set Good(W) from the properties of one good partial resolution.

In Section 4.1, we construct a root system $\mathfrak{R}$—and hence the corresponding root lattice $\mathfrak{L}$—from the codimension 2 singularities of W and relate the root lattice to the geometry of good partial resolutions. The root system $\mathfrak{R}$ is the co-root system of the semisimple part of the gauge algebra. In Section 4.2, we construct a root system $\mathfrak{R}^p$ (and hence the corresponding root lattice $\mathfrak{L}^p$) at each closed, singular point p of W, and, for each good partial resolution π, a homomorphism ϕ_π^p from $\mathfrak{L} \otimes \mathbb{Q}$ to $\mathfrak{L}^p \otimes \mathbb{Q}$. This *enhancement homomorphism* gives rise to a set $\mathcal{W}_\pi^p$ of *enhancement weights* of $\mathfrak{L}$. The set $\mathcal{W}_\pi^p$ depends *à priori* on choice of good partial resolution π, but our main results—which we state in Section 5—provide sufficient conditions for $\mathcal{W}_\pi^p$ to be independent of π and derive consequences for the structure of Good(W).

4.1 Co-root system of the gauge algebra

In Situation 4.1, we construct a lattice $\mathfrak{L}_i$ for each η_i and relate the lattices to the Picard groups of good partial resolutions. The lattice $\mathfrak{L}_i$ will be defined to be the root lattice of the root system corresponding to the Du Val singularity type of W at η_i. To make sense of this construction and to relate W to the geometry of good partial resolutions, we need the following proposition.

Proposition 4.4. *In Situation 4.1, let $1 \leq i \leq s$ be arbitrary. The following conclusions hold.*

(a) The local ring $\mathcal{O}_{W,\eta_i}$ has a Du Val singularity.

(b) For any crepant partial resolution $\pi : Y \to W$ with Y nonsingular in codimension 2, the base-change $\pi \times_W \operatorname{Spec} \mathcal{O}_{W,\eta_i} : Y \times_W \operatorname{Spec} \mathcal{O}_{W,\eta_i} \to \operatorname{Spec} \mathcal{O}_{W,\eta_i}$ is the minimal resolution of $\operatorname{Spec} \mathcal{O}_{W,\eta_i}$.

Proof. We first prove Part (a). As normality is preserved under localization [76, Tag 00GY], the local ring $\mathcal{O}_{W,\eta_i}$ is normal. By Proposition 3.26, W has rational Gorenstein singularities. The stalks of Gorenstein schemes (resp. schemes with rational singularities) are by definition Gorenstein (resp. have rational singularities). Hence, the local ring $\mathcal{O}_{W,\eta_i}$ has a Du Val singularity.

We next prove Part (b). As Y is nonsingular in codimension 2, Y has only finitely many singular points. Let Y_{sing} denote the singular locus of Y. Define $W^0 = W \setminus \pi(Y_{\text{sing}})$ and $Y^0 = \pi^{-1}(W^0)$, which are open subschemes of W and Y, respectively. Note that Y^0 is nonsingular, hence in particular normal and Gorenstein. As Y^0 and W^0 are normal, Gorenstein, and quasi-projective, Proposition 3.12 hence implies that Y^0/W^0 has an invertible dualizing sheaf whose class in $\operatorname{Pic} Y^0$ is torsion.

By Proposition 3.13, $Y \times_W \operatorname{Spec} \mathcal{O}_{W,\eta_i} / \operatorname{Spec} \mathcal{O}_{W,\eta_i}$ has an invertible dualizing sheaf whose class in $\operatorname{Pic} Y \times_W \operatorname{Spec} \mathcal{O}_{W,\eta_i} / \operatorname{Spec} \mathcal{O}_{W,\eta_i}$. Proposition 3.16(b) implies that $Y \times_W \operatorname{Spec} \mathcal{O}_{W,\eta_i}$ must be the minimal resolution of $\operatorname{Spec} \mathcal{O}_{W,\eta_i}$. □

In Situation 4.1, for $1 \leq i \leq s$, let $\widetilde{\pi}_i : \widetilde{Y}^{\eta_i} \to X$ denote the minimal resolution of $\operatorname{Spec} \mathcal{O}_{W,\eta_i}$, which exists by Theorem 3.15 and Proposition 4.4(a). Denote the exceptional curves of $\widetilde{\pi}_i$ by $\mathtt{C}_i^{(1)}, \ldots, \mathtt{C}_i^{(s)}$. Let $\mathfrak{R}_i = \mathfrak{R}(\mathcal{O}_{W,\eta_i})$ denote the root system associated to the Du Val singularity $\mathcal{O}_{W,\eta_i}$. Let $\Delta_i = \{\mathbf{c}_i^{(1)}, \ldots, \mathbf{c}_i^{(k_i)}\}$ be the given basis of simple roots of $\mathfrak{L}_i = \mathfrak{L}(\mathcal{O}_{W,\eta_i})$, where $\mathbf{c}_i^{(j)}$ is the basis vector corresponding to $\mathtt{C}_i^{(j)}$. Let $\mathfrak{L} = \bigoplus_{i=1}^s \mathfrak{L}_i$, which we equip with the bilinear form $\langle -, - \rangle = \bigoplus_{i=1}^s \langle -, - \rangle_{\mathcal{O}_{W,\eta_i}}$.

The lattice $\mathfrak{L}_i$ reflects the singularity type of W along the curve Z_i. From the perspective of F-theory, $\mathfrak{L}_i$ is the co-root lattice of the local gauge factor along the curve Z_i, and $\mathfrak{L}$ is the co-root lattice of the semisimple part of the gauge group. We can therefore define a gauge algebra as in the F-theory literature.

Definition 4.5. In Situation 4.1, the *gauge algebra* $\mathfrak{g}$ is a complex semisimple Lie algebra with co-root system isomorphic to $\mathfrak{R}$.

Remark 4.6. As $\mathfrak{R}$ is the co-root system of $\mathfrak{g}$, the types of $\mathfrak{g}$ and $\mathfrak{R}$ are Langlands dual. Specifically, each irreducible factor of $\mathfrak{R}$ of type B_n (resp. C_n) corresponds to a simple factor of $\mathfrak{g}$ of type C_n (resp. B_n). This appearance of the Langlands dual arises in the F-theory literature as well (see Witten [84], Morrison and Vafa [66, 67], Vafa [79], Intriligator et al. [35], and Morrison and Seiberg [63]).

We next relate $\mathfrak{L}$ to the geometry of crepant partial resolutions of W. Let $\pi : Y \to W$ be a crepant partial resolution of W with Y nonsingular in codimension 2. Proposition 4.4(b) guarantees that $\pi \times_W \operatorname{Spec} \mathcal{O}_{W,\eta_i}$ is the minimal resolution of $\operatorname{Spec} \mathcal{O}_{W,\eta_i}$. As a result, we can (canonically) identify $\pi^{-1}(\eta_i)$ with $\widetilde{\pi}_i^{-1}(\eta_i)$ for

$1 \leq i \leq s$. For $1 \leq i \leq s$ and $1 \leq j \leq k_i$, let $\eta_i^{(j)}$ be the generic point of $\mathtt{C}_i^{(j)}$ and let $Z_i^{(j)} = \overline{\{\eta_i^{(j)}\}}$ be the subvariety of Y with generic point $\eta_i^{(j)}$. In accordance with the F-theory literature, we call the varieties $Z_i^{(j)}$ the *Cartan divisors.*

To relate $\mathfrak{L}$ to the Picard group, we need to specialize to good partial resolutions. Let $\pi : Y \to W$ be a good partial resolution. By Proposition 3.30, Y is nonsingular in codimension 2, and hence the discussion of the previous paragraph applies. Define a homomorphism $\psi_\pi : \mathfrak{L} \to \operatorname{Pic} Y \otimes \mathbb{Q}$ by $\psi_\pi(\mathbf{c}_i^{(j)}) = c_{1,\mathbb{Q}}^{-1}([Z_i^{(j)}])$, where

$$c_{1,\mathbb{Q}} = c_1 \otimes \mathbb{Q} : \operatorname{Pic} Y \otimes \mathbb{Q} \to \operatorname{Cl} Y \otimes \mathbb{Q}$$

is an isomorphism because Y is $\mathbb{Q}$-factorial. Identifying $\operatorname{Pic} Y \otimes \mathbb{Q}$ with $\operatorname{Pic} Y' \otimes \mathbb{Q}$ for any good partial resolution $\pi' : Y' \to W$ by taking proper transforms, note that ψ_π is independent of π by construction. We therefore write ψ for the homomorphism ψ_π for any $\pi \in \operatorname{Good}(W)$.

4.2 The enhancement, and obtaining a set of weights

Let $p \in W_{\text{sing}}$ be a closed point. To define the enhancement, we first construct a root system that captures the singularity type of W at p. Let $S_p = \{i \,|\, p \in Z_i\}$ denote the set of indices i such that Z_i passes through p.

We fix once and for all a hyperplane section H^p through $p \in W$ such that $\mathcal{O}_{H^p,p}$ has a Du Val singularity. Such a hyperplane section exists because W has cDV singularities. In light of Proposition 3.24, we can assume that H^p meets Z_i dimensionally transversely at p for all $i \in S_p$. Let $\widetilde{\pi}^p : \widetilde{Y}^p \to \operatorname{Spec} \mathcal{O}_{H^p,p}$ denote the minimal resolution of $\operatorname{Spec} \mathcal{O}_{H^p,p}$, which exists by Theorem 3.15.

We consider the root system and root lattice associated to the local ring $\mathcal{O}_{H^p,p}$, which has a Du Val singularity. Specifically, let $\mathfrak{L}^p = \mathfrak{L}(\mathcal{O}_{H^p,p})$, denote the basis vectors by $\mathbf{r}_1^p, \ldots, \mathbf{r}_{k^p}^p$, and let $C_1^p, \ldots, C_{k^p}^p$ denote the corresponding exceptional curves of $\widetilde{\pi}^p$. We equip $\mathfrak{L}^p$ with the bilinear form $\langle -, - \rangle_{\mathcal{O}_{H^p,p}}$, which we denote by $\langle -, - \rangle^p$. Let $\mathfrak{R}^p = \mathfrak{R}(\mathcal{O}_{H^p,p})$ and let $\mathfrak{R}^{p,+}$ denote the set of positive roots of $\mathfrak{R}^p$ (for any polarization for which $\mathbf{r}_1^p, \ldots, \mathbf{r}_{k^p}^p$ are the simple roots).

To define and work with the enhancement homomorphism, we consider a more special situation than Situation 4.1 by fixing p and a good partial resolution.

Situation 4.7. In Situation 4.1, let $p \in W_{\text{sing}}$ be a closed point and let $\pi : Y \to W$ be a good partial resolution.

In Situation 4.7, Theorem 3.27 implies that $\widetilde{\pi}^p$ factors via $\pi \times_W \operatorname{Spec} \mathcal{O}_{H^p,p}$, say via a morphism $\Lambda_\pi^p : \widetilde{Y}^p \to Y \times_W \operatorname{Spec} \mathcal{O}_{H^p,p}$. Let $\Phi_\pi^p : \widetilde{Y}^p \to Y$ denote the composite $\Phi_\pi^p = \Gamma_\pi^p \circ \Lambda_\pi^p$, where $\Gamma_\pi^p : Y \times_W \operatorname{Spec} \mathcal{O}_{H^p,p} \to Y$ is the natural morphism. We use the pullback $(\Phi_\pi^p)^* : \operatorname{Pic} Y \to \operatorname{Pic} \widetilde{Y}^p$ to construct the enhancement homomorphism.

Definition 4.8. In Situation 4.7, let $\pi : Y \to W$ be a good partial resolution and let $p \in W_{\text{sing}}$ be a closed point. The *enhancement homomorphism* $\phi_\pi^p : L \to \mathfrak{L}^p \otimes \mathbb{Q}$ is defined by

$$\phi_\pi^p = \theta_{\mathcal{O}_{H^p,p},\mathbb{Q}}^{-1} \circ (\Phi_\pi^p)_{\mathbb{Q}}^* \circ \psi,$$

where $\theta_{\mathcal{O}_{H^p,p},\mathbb{Q}}^{-1} : \operatorname{Pic} \widetilde{Y}^p \otimes \mathbb{Q} \to \mathfrak{L}^p \otimes \mathbb{Q}$ was defined in Section 3.4 using Proposition 3.18 and $(\Phi_\pi^p)_{\mathbb{Q}}^* = (\Phi_\pi^p)^* \otimes \mathbb{Q} : \operatorname{Pic} Y \otimes \mathbb{Q} \to \operatorname{Pic} \widetilde{Y}^p \otimes \mathbb{Q}$.

We use ϕ_π^p to obtain a set of weights. We first recall the definition of weights. Formally, the *weight group* of $\mathfrak{R}$ is the lattice-theoretic dual

$$\mathfrak{L}^\dagger(\mathfrak{R}) = \{\lambda \in E \mid \langle \lambda, \alpha \rangle \in \mathbb{Z} \text{ for all } \alpha \in \mathfrak{L}(\mathfrak{R})\}.$$

We call the elements of the weight group *weights*. All roots are weights because the inner products of roots are integral. Note that $\mathfrak{L}^\dagger(\mathfrak{R})$ is the set of integral weights for the gauge algebra $\mathfrak{g}$ by construction.

To obtain a set of enhancement weights, we consider the image of the set of roots $\mathfrak{R}^p$ under the adjoint of the enhancement homomorphism.

Definition 4.9. In Situation 4.7, let $(\phi_\pi^p)^\dagger : \mathfrak{L}^p \otimes \mathbb{Q} \to \mathfrak{L}_p \otimes \mathbb{Q}$ denote the adjoint of ϕ_π^p with respect to the inner products $\langle -, - \rangle^p$ and $\langle -, - \rangle$. The multiset of *enhancement weights* is the image $\mathcal{W}_\pi^p$ of $\mathfrak{R}^p$ under $(\phi_\pi^p)^\dagger$, and the multiset of *positive enhancement weights* is the image $\mathcal{W}_\pi^{p,+}$ of $\mathfrak{R}^{p,+}$ under $(\phi_\pi^p)^\dagger$. We denote by $\mathcal{W}_{\pi,\neq 0}^p$ (resp. $\mathcal{W}_{\pi,\neq 0}^{p,+}$) the multiset of nonzero elements of $\mathcal{W}_\pi^p$ (resp. $\mathcal{W}_\pi^{p,+}$).

Let $\mathfrak{L}_p = \bigoplus_{i \in S_p} \mathfrak{L}_i \subseteq \mathfrak{L}$ denote the portion of $\mathfrak{L}$ associated to curves in the singular locus that pass through p, and let $\mathfrak{R}_p = \bigoplus_{i \in S_p} \mathfrak{R}_i \subseteq \mathfrak{L}_p$ denote the set of roots that come from curves that pass through p. To understand the interaction between the enhancement homomorphism and the forms $\langle -, - \rangle^p$ and $\langle -, - \rangle$, we need to require that the singular curves are nonsingular at p and the Cartan divisors $Z_i^{(j)}$ are Cartier. The former condition was expected by the F-theory literature to be needed to obtain (quasi-)minuscule matter representations (see Klevers et al. [44]). The latter condition is a mild regularity condition on Y.

Proposition 4.10. *In Situation 4.7, suppose that Z_i is nonsingular at p for all $i \in S_p$ and that $[Z_i^{(j)}]$ is Cartier for all $i \in S_p$ and $1 \leq j \leq k_i$. The homomorphism ϕ_π^p factors through $\mathfrak{L}_p$ and induces an isometric embedding of $\mathfrak{L}_p$ into $\mathfrak{L}^p$. In particular, we have that $\mathbf{w} \in \mathfrak{L}^\dagger(\mathfrak{R}_p)$ and that $\langle \mathbf{w}, \mathbf{w} \rangle \leq 2$ for all $\mathbf{w} \in \mathcal{W}_\pi^p$.*

Proposition 4.10, which we prove in Section 6, provides conditions under which the enhancement homomorphism gives rise to an isometric embedding of $\mathfrak{L}_p$ into $\mathfrak{L}^p$. In this case, the enhancement weights have norm at most $\sqrt{2}$ with respect to $\langle -, - \rangle$ and only have nonzero inner product with the roots that lie in $\mathfrak{R}_p$.

When furthermore the root system $\mathfrak{R}$ is simply-laced, the set of weights acquires a representation-theoretic interpretation. We say that a nonzero weight $\mathbf{w} \in \mathfrak{L}^\dagger(\mathfrak{R})$ is *(quasi-)minuscule* if the orbit $\mathfrak{W}(\mathfrak{R})\mathbf{w}$ is the set of nonzero weights of a finite-dimensional representation of the gauge algebra $\mathfrak{g}$. We denote by $\text{Q-Minusc}(\mathfrak{R}) \subseteq \mathfrak{L}^\dagger(\mathfrak{R})$ the set of (quasi-)minuscule weights. Likewise, a representation $\mathbf{R}$ of a complex semisimple Lie algebra $\mathfrak{g}$ is *(quasi-)minuscule* if its set of nonzero weights forms a single orbit for the action of the Weyl group. (By construction, a dominant weight $\mathbf{w}$ is (quasi-)minuscule if and only if the representation with highest weight $\mathbf{w}$ is (quasi-)minuscule.)

Proposition 4.11. *In Situation 4.7, suppose that Z_i is nonsingular at p for all $i \in S_p$. Suppose furthermore that $[Z_i^{(j)}]$ is Cartier for all $i \in S_p$ and $1 \leq j \leq k_i$, and that $\mathfrak{R}_i$ is simply-laced for all $i \in S_p$. The following conclusions hold.*

(a) We have that $\mathcal{W}_{\pi,\neq 0}^p \subseteq \text{Q-Minusc}(\mathfrak{R})$.

(b) If $w \in \mathfrak{W}(\mathfrak{R})$, then we have that $w(\mathcal{W}_\pi^p) = \mathcal{W}_\pi^p$ as multisets.

Proposition 4.11, which we prove in Section 6, implies the existence of matter representations for simply-laced semi-simple gauge algebras.

Definition 4.12. Under the hypotheses of Proposition 4.11(b), the *matter representation* $\mathbf{R}_\pi^p$ is a representation of the gauge algebra $\mathfrak{g}$ whose multiset of nonzero weights is the multiset $\mathcal{W}_{\pi,\neq 0}^p$ of nonzero enhancement weights.

Under the hypotheses of Proposition 4.11(b), Proposition 4.11 implies the existence of a matter representation. The matter representation is unique up to isomorphism and trivial factors by construction. It is a direct sum of (quasi-)minuscule representations by Proposition 4.11(a) and is self-dual because we have that $-\mathcal{W}_\pi^p = \mathcal{W}_\pi^p$ as multisets.

Note that our definition of the matter representation comes equipped with well-defined multiplicities for every non-trivial irreducible factor. As our matter representations are associated to singular points, they are closest to the "local" matter representations of F-theory.[36] When a representation appears at only finitely points, it would be interesting to see if the multiplicities in our matter representations match the multiplicities that are predicted by anomaly cancellation for the Intrilegator–Morrison–Seiberg [35] superpotential.

Our definition of the matter representation is similar in spirit to the Katz–Vafa [37] predictions of matter representations from F-theory (see also Grassi and Morrison [27, 28]). Indeed, both we and they construct matter representations by restricting by considering the Du Val singularity type of an enhancement. However, we consider a map of root lattices, which Katz and Vafa [37] have assumed to come from a particular map between Lie groups. Moreover, the Katz–Vafa predictions are only valid when components of the discriminant locus meet transversely, whereas we do not need to make any such restriction.

5 Statements of the main results

We are now ready to state our main results, in which we characterize the KKMR decomposition of a good partial decomposition in terms of the gauge algebra, the enhancement weights, and the matter representation. Section 5.1 describes the regularity assumptions that we make. Section 5.2 describes our main results for general gauge algebras, and Section 5.3 specializes to the case of simply-laced gauge algebras.

5.1 Setup

We specialize Situation 4.1 by imposing regularity conditions to study the set of good partial resolutions. We first require that W is $\mathbb{Q}$-factorial. As we show in Proposition 8.6 in Section 8, this requirement is satisfied by the Weierstrass models of elliptic fibrations with Mordell-Weil rank 0. Such elliptic fibrations are the ones whose gauge algebras are semisimple in F-theory [52, 59, 65].

Under $\mathbb{Q}$-factoriality, there is a particularly simple characterization of good partial

[36]Nevertheless, if a finite-dimensional irreducible representation appears in local matter representations at infinitely many points, it becomes non-localized.

resolutions in terms of Cartan divisors. Although we do not use this characterization in our main arguments, we nevertheless present it to make the concept of a good partial resolution more concrete.

Proposition 5.1. *In Situation 4.1, suppose that* W *is* $\mathbb{Q}$*-factorial and let* $\pi : Y \to W$ *be a projective, crepant partial resolution. If* Y *is nonsingular in codimension 2 and all of the Cartan divisor classes* $[Z_i^{(j)}]$ *are* $\mathbb{Q}$*-Cartier, then* π *is good.*

Proof. Corollary 3.33 implies that Y has terminal singularities. It remains to prove that Y is $\mathbb{Q}$-factorial. Note that Y is normal by the definition of a partial resolution.

Let Z be a prime divisor on Y. As W is $\mathbb{Q}$-factorial, there exists $n \in \mathbb{Z}_{>0}$ such that $n\pi_*[Z]$ is Cartier. Hence, $\pi^*(n\pi_*[Z])$ is well-defined and is a Cartier divisor on Y. By construction, the divisor $n[Z] - \pi^*(n\pi_* Z)$ is supported on the exceptional locus of π, and hence can be expressed as a $\mathbb{Z}$-linear combination of the classes $[Z_i^{(j)}]$ for $1 \leq i \leq s$ and $1 \leq j \leq k_i$. Because the classes $[Z_i^{(j)}]$ are Cartier, we have that $n[Z]$ is Cartier. As Z was arbitrary, we have proven that Y is $\mathbb{Q}$-factorial, as desired. □

We also impose a condition on good partial resolutions—we instead consider very good partial resolutions.

Definition 5.2. In Situation 4.1, a good partial resolution $\pi : Y \to W$ is *very good* if $[Z_i^{(j)}]$ is Cartier for all $1 \leq i \leq s$ and $1 \leq j \leq k_i$, and each closed point $p \in W$ satisfies at least one of the following conditions:

(i) Y is nonsingular above p; or

(ii) all enhancement weights above p are proportional to roots: $\mathcal{W}_\pi^p \subseteq \mathbb{R}_{\geq 0}\mathfrak{R}$.

Intuitively, a good partial resolution is very good if the Cartan divisors are Cartier and the enhancement weights below singular points are all proportional to roots. The very goodness of good partial resolutions turns out to be a property of W—independent of the choice of good partial resolution—as long as W is $\mathbb{Q}$-factorial.

Proposition 5.3. *In Situation 4.1, if* W *is* $\mathbb{Q}$*-factorial and there exists a very good partial resolution, then every good partial resolution is very good.*

Proposition 5.3, which we prove in Section 6, implies that if we assume that one good partial resolution of a $\mathbb{Q}$-factorial threefold with cDV singularities is very good, then it is guaranteed that every good partial resolution is very good. We make this assumption—in addition to the assumptions of $\mathbb{Q}$-factoriality of W and the nonsingularity of the irreducible components of W_{sing}—to specialize Situation 4.1.

Situation 5.4. In Situation 4.1, suppose that Z_i is nonsingular for all $1 \leq i \leq s$, and that W is $\mathbb{Q}$-factorial and admits a very good partial resolution $\pi : Y \to W$.

In Situation 5.4, it follows from Proposition 5.3 that the good partial resolutions of W are precisely the very good partial resolutions of W, so that $\mathrm{Good}(W)$ is the set of isomorphism classes of very good partial resolutions.

5.2 Results for general gauge algebras

Our first result characterizes the movable, nef, and ample cones in terms of the gauge algebra and the enhancement weights. The result then applies these characterizations to show that few hyperplanes can be the spans of codimension 1 faces in the KKMR decomposition, and to provide an effective bound on the number of (very) good partial resolutions.

Theorem 5.5. *In Situation 5.4, the following conclusions hold.*

(a) The homomorphism $\psi_{\mathbb{R}} : \psi \otimes \mathbb{R}$ induces an isomorphism from $\mathfrak{L} \otimes \mathbb{R}$ to $N^1(Y/W)$, and we have that

$$\begin{aligned}\psi_{\mathbb{R}}^{-1}(\mathrm{Mov}(Y/W)) &= \psi_{\mathbb{R}}^{-1}(\overline{\mathrm{Mov}}(Y/W)) \\ &= \{\mathbf{c} \in \mathfrak{L} \otimes \mathbb{R} \,|\, \langle \mathbf{c}, \mathbf{c}_i^{(j)} \rangle \leq 0 \text{ for all } 1 \leq i \leq s \text{ and } 1 \leq j \leq k_i\}.\end{aligned}$$

(b) For all crepant resolutions $\pi' : Y' \to W$, we have that

$$\psi_{\mathbb{R}}^{-1}(\mathrm{Amp}(Y'/W)) = \left\{ \mathbf{c} \in \mathfrak{L} \otimes \mathbb{R} \;\middle|\; \langle \mathbf{c}, \mathbf{w} \rangle < 0 \text{ for all } \mathbf{w} \in \bigcup_{p \in W_{\mathrm{sing}}(\mathbb{C})} \mathcal{W}_{\pi', \neq 0}^{p,+} \right\}$$

$$\psi_{\mathbb{R}}^{-1}(\overline{\mathrm{Amp}}(Y'/W)) = \left\{ \mathbf{c} \in \mathfrak{L} \otimes \mathbb{R} \;\middle|\; \langle \mathbf{c}, \mathbf{w} \rangle \leq 0 \text{ for all } \mathbf{w} \in \bigcup_{p \in W_{\mathrm{sing}}(\mathbb{C})} \mathcal{W}_{\pi'}^{p,+} \right\}.$$

(c) There is a finite, polyhedral decomposition

$$\psi_{R}^{-1}(\mathrm{Mov}(Y/W)) = \bigcup_{(\pi' : Y' \to W) \in \mathrm{Good}(W)} \psi_{\mathbb{R}}^{-1}(\overline{\mathrm{Amp}}(Y'/W)), \tag{1}$$

where the cones $\psi_{\mathbb{R}}^{-1}(\overline{\mathrm{Amp}}(Y/W))$ are disjoint in their interior. Each codimension 1 face spans a hyperplane that is normal to a weight $\mathbf{w}$ with $\langle \mathbf{w}, \mathbf{w} \rangle \leq 2$.

(d) We have that

$$|\,\mathrm{Good}(W)| \leq 2^{2^{-1+2\sum_{i=1}^{s} k_i}}.$$

We prove Theorem 5.5 in Section 6. Each of the parts has a different interpretation. Theorem 5.5(a) implies that the homomorphism $\psi_{\mathbb{R}}$ identifies the closed dual fundamental Weyl chamber of the gauge algebra with the movable cone. Theorem 5.5(b) shows that $\psi_{\mathbb{R}}^{-1}$ identifies the nef (resp. ample) cone with the cone of combinations of roots with which all of the positive enhancement weights are nonnegative (resp. positive) inner product. Theorem 5.5(c) proves that the KKMR decomposition is globally polyhedral and shows that very few hyperplanes—only the normals to integral weights of length at most $\sqrt{2}$—can appear as the spans of codimension 1 faces. Theorem 5.5(d) concludes that the number of (very) good partial resolutions is at most a double exponential in the rank of the gauge algebra.

Remark 5.6. Kawamata and Matsuki [40] have shown that every threefold with canonical singularities has only finitely many projective crepant partial resolutions. Theorem 5.5(d) provides an effective version of their result in a case.

If $\mathfrak{R}$ is simple of type C_n for $n \geq 9$, E_8, F_4, or G_2, then every integral weight of length at most $\sqrt{2}$ is proportional to a root. As a result, the only hyperplanes that can appear in the KKMR decomposition are the walls of the dual fundamental Weyl chamber, and hence there can be at most one chamber in the KKMR decomposition. This property implies that there is only one (very) good partial resolution.

Corollary 5.7. *In Situation 5.4, if W_{sing} has a unique positive-dimensional irreducible component at whose generic point W has a Du Val singularity of type C_n for $n \geq 9$, E_8, F_4, or G_2, then W has a unique good partial resolution.*

Proof. The hypothesis of the corollary requires that $\mathfrak{R}$ is irreducible of type C_n for $n \geq 9$, E_8, F_4, or G_2. By Theorem 5.5(c), it suffices to show that every weight of length at most $\sqrt{2}$ is proportional to an element of $\mathfrak{R}$.

In the cases of E_8, F_4, and G_2, the Cartan matrix has determinant 1 [6, Planches VII–IX], and hence we have that $\mathfrak{L}^{\dagger}(\mathfrak{R}) = \mathfrak{L}(\mathfrak{R})$. As $\mathfrak{L}(R)$ is an even lattice,[37] we

[37]It follows from Theorem 2.2 that root lattices are even, in the sense that all elements have square-lengths that are even integers.

must have that $\langle \mathbf{w}, \mathbf{w} \rangle \geq 2$ for all nonzero $\mathbf{w} \in \mathfrak{L}^\dagger(\mathfrak{R})$. By Proposition 2.4, every weight $\mathbf{w} \in \mathfrak{L}^\dagger(\mathfrak{R}) = \mathfrak{L}(\mathfrak{R})$ with $\langle \mathbf{w}, \mathbf{w} \rangle = 2$ must be a root. Therefore, any weight of length at most $\sqrt{2}$ must be proportional to an element of $\mathfrak{R}$.

We now consider the case of type C_n for $n \geq 9$. The standard formulae for the inverses of the Cartan matrices of irreducible root systems [55, 83] imply that the lengths of all fundamental weights are at least 1, and the lengths of all fundamental weights except the first and second are greater than $\sqrt{2}$. Hence, the only dominant weights of length at most $\sqrt{2}$ are the first and second fundamental weights. The first fundamental weight is proportional to a root while the second fundamental weight is the highest root. Hence, every weight of length at most $\sqrt{2}$ must be proportional to an element of $\mathfrak{R}$. □

Remark 5.8. The hypothesis of Corollary 5.7 amounts to assuming that the gauge algebra $\mathfrak{g}$ is isomorphic to one of $\mathfrak{so}_{2n+1}$ for $n \geq 9$, $\mathfrak{e}_8$, $\mathfrak{f}_4$, or $\mathfrak{g}_2$, because the type of $\mathfrak{g}$ is Langlands dual to the type of $\mathfrak{R}$. For the case of $\mathfrak{so}_{2n+1}$, the hypothesis that $n \geq 9$ is needed because weights of the spin representation have norm less than or equal to $\sqrt{2}$ for $n \leq 8$. These weights are not proportional to roots, and hence can give rise to nontrivial KKMR decompositions.

The enhancement weights might *à priori* depend on the choice of a (very) good partial resolution, making it difficult to determine the KKMR decomposition without computing *all* of the (very) good partial resolutions. When the set of enhancement weights is known to be independent of phase, it suffices to compute *one* (very) good partial resolution to determine the KKMR decomposition. Our next result shows that as long as every enhancement weight of length at most $\sqrt{\frac{8}{9}}$ is proportional to a root, the multisets of enhancement weights are independent of the choice of (very) good partial resolution. The result also describes the consequences of this independence property for the KKMR decomposition.

Theorem 5.9. *In Situation 5.4, if each* $\mathbf{w} \in \bigcup_{p \in W_{\mathrm{sing}}(\mathbb{C})} \mathcal{W}_\pi^p$ *with* $\langle \mathbf{w}, \mathbf{w} \rangle \leq \frac{8}{9}$ *is proportional to an element of* $\mathfrak{R}$*, then the following conclusions hold.*

(a) We have that $\mathcal{W}_{\pi'}^p = \mathcal{W}_\pi^p$ *as multisets for all* $\pi' \in \mathrm{Good}(W)$*.*

(b) The KKMR decomposition (1) *given in Theorem 5.5(c) coincides with the decomposition of* $\psi_{\mathbb{R}}^{-1}(\mathrm{Mov}(Y/W))$ *into closed chambers for the hyperplane ar-*

rangement consisting of the hyperplanes that are normal to elements of

$$\bigcup_{p \in W_{\mathrm{sing}}(\mathbb{C})} \mathcal{W}^p_{\pi, \neq 0}.$$

We prove Theorem 5.9 in Section 7. Theorem 5.9(a) states our result on the resolution-independence of the enhancement weights. Theorem 5.9(b) shows that the KKMR decomposition is given by the decomposition of the Weyl chamber into chambers for the hyperplane arrangement with hyperplanes normal to the enhancement weights of any given (very) good partial resolution.

Theorem 5.9(b) resembles results of Brieskorn [9] and Matsuki [57]. Brieskorn [9] has shown an analogous result for simultaneous resolutions of families of Du Val singularities. Our result applies to a larger class of cDV singularities and does not require that every hyperplane section with a cDV singularity is resolved. Nevertheless, our result is not strictly stronger than that of Brieskorn [9] due to our assumption on very short enhancement weights being proportional to roots. Matsuki [57, Theorem II-2-1-1] has shown an analogous result for elliptic threefolds with Kodaira fibers.[38] However, non-Kodaira fibers can arise in codimension 2 in elliptic fibrations. As we show in Section 8, our result can apply even in the presence of non-Kodaira fibers.

For most gauge algebras, every integral weight of length at most $\sqrt{\frac{8}{9}}$ is proportional to a root. Indeed, this condition can only fail when one of the root systems $\mathfrak{R}_i$ is of type A_n for $2 \leq n \leq 8$ or of type C_3. When none of those root systems arise as an $\mathfrak{R}_i$, the conclusions of Theorem 5.9 are guaranteed to hold in Situation 5.4.

Corollary 5.10. *In Situation 5.4, if the Du Val singularity type of W at every codimension 2 point of W_{sing} is not A_n for $2 \leq n \leq 8$ or C_3, then the hypothesis of Theorem 5.9 is satisfied. In particular, the conclusions of Theorem 5.9 hold.*

Proof. The hypothesis of the corollary requires that, for all $1 \leq i \leq s$, the root system $\mathfrak{R}_i$ is not of type A_n for $2 \leq n \leq 8$ or of type C_3. We first show that $\langle \mathbf{w}, \mathbf{w} \rangle \geq \frac{1}{2}$ for all $\mathbf{w} \in \mathfrak{L}^\dagger(\mathfrak{R}_i)$ and that $\langle \mathbf{w}, \mathbf{w} \rangle > \frac{8}{9}$ when $\mathbf{w}$ is not proportional to any element of $\mathfrak{R}_i$. To prove this claim, we divide into cases based on the type of $\mathfrak{R}_i$, excluding the cases of types A_1 and B_2, which can be dealt with through simple explicit computations.

[38]Technically, Matsuki [57] uses a lemma of Burns and Rapoport [10] on how the restrictions of divisors to nonsingular surfaces change during flops between threefolds. In the presence of non-Kodaira fibers, some hyperplane sections are not fully resolved in (very) good partial resolutions, and hence the Burns–Rapoport Lemma does not apply.

Case 1: $\mathfrak{R}_i$ is of type B_n for $n \geq 4$, of type D_n for $n \geq 4$, or of type E_6, E_7, E_8, F_4, or G_2. The standard formulae for the inverses of the Cartan matrices of irreducible root systems [55, 83] imply that the lengths of all fundamental weights are at least 1. As a result, all dominant weights (and hence all nonzero weights) have length at least 1.

Case 2: $\mathfrak{R}_i$ is of type A_n for $n \geq 9$. The standard formulae for the inverses of the Cartan matrices of irreducible root systems [55, 83] imply that the lengths of all fundamental weights but the first and last are greater than 1, and that the first and last fundamental weights have length $\sqrt{\frac{n}{n+1}}$. As we have assumed that $n \geq 9$, all dominant weights (and hence all nonzero weights) have length at least $\sqrt{\frac{9}{10}}$.

Case 3: $\mathfrak{R}_i$ is of type C_n for $n \geq 3$. The standard formulae for the inverses of the Cartan matrices of irreducible root systems [55, 83] imply that the lengths of all fundamental weights but the first are greater than 1. Therefore, the only dominant weight of length less than 1 is the first fundamental weight, which in turn has length $\frac{1}{\sqrt{2}}$ and is proportional to the highest root. Hence, every dominant weight other than the first fundamental weight has length at least 1. As a result, all nonzero weights other than the weights of the first fundamental representation—whose weights are proportional to roots—have length at least 1.

By Theorem 2.2, the cases exhaust all possibilities, and hence we have proven the claim. Taking linear combinations, it follows that $\langle \mathbf{w}, \mathbf{w} \rangle > \frac{8}{9}$ for all weights $\mathbf{w}$ that are not proportional to elements of $\mathfrak{R}$. The corollary follows by Theorem 5.9. □

Remark 5.11. The hypothesis of Corollary 5.10 amounts to assuming that the gauge algebra $\mathfrak{g}$ does not have a simple factor that is isomorphic to $\mathfrak{sl}_{n+1}$ for $2 \leq n \leq 9$ or $\mathfrak{so}_7$. The difficulty with $\mathfrak{sl}_{n+1}$ for $2 \leq n \leq 9$ (resp. $\mathfrak{so}_7$) is that the weights of the first fundamental (resp. spin) representation have length less than $\sqrt{\frac{8}{9}}$ and are not proportional to roots.

5.3 Results for simply-laced gauge algebras

In light of Proposition 4.11, the enhancement weights come from a matter representation when the gauge algebra is simply-laced. In this case, the hyperplane arrangement of hyperplanes that are normal to the nonzero enhancement weights takes the form of the singular locus of the Intrilegator–Morrison–Seiberg superpotential from the physics literature. Esole et al. [18, 20] have provided a mathematical formalization.

Definition 5.12 ([18, Definition 1.1])**.** Let $\mathfrak{g}$ be a complex, semisimple Lie algebra, let $\mathfrak{h}$ be a split, real form of a Cartan subalgebra of $\mathfrak{g}$, and let $\mathbf{R}$ be a locally finite representation of $\mathfrak{g}$ with only finitely many non-isomorphic irreducible factors. We denote by $\mathrm{I}(\mathfrak{g}, \mathbf{R})$ the decomposition of the dual fundamental Weyl chamber in $\mathfrak{h}$ into the chambers for the hyperplane arrangement consisting of the hyperplanes that are orthogonal to the nonzero weights of $\mathbf{R}$.

We can reinterpret Corollary 5.10 in the case of simply-laced gauge algebras using the concept of a matter representation and the notation $\mathrm{I}(\mathfrak{g}, \mathbf{R})$.

Corollary 5.13. *In Situation 5.4, if $\mathfrak{R}$ is simply-laced and $\mathfrak{R}_i$ is not of type A_n for $2 \leq n \leq 8$ for any i, then the following conclusions hold:*

(a) For all closed points $p \in W_{\mathrm{sing}}$, the matter representation $\mathbf{R}^p_\pi$ is independent (up to isomorphism and trivial factors) of the choice of $\pi \in \mathrm{Good}(W)$.

(b) The KKMR decomposition (1) *is of the form* $\mathrm{I}(\mathfrak{g}, \mathbf{R})$*, where $\mathfrak{g}$ is the gauge of algebra and $\mathbf{R} = \bigoplus_{p \in W_{\mathrm{sing}}(\mathbb{C})} \mathbf{R}^p_\pi$ (which is independent up to isomorphism and trivial factors of the choice of $\pi \in \mathrm{Good}(W)$ by Part (a)).*

(c) Every irreducible factor of $\mathbf{R}$ is a (quasi-)minuscule representation whose highest weight has length at most $\sqrt{2}$.

(d) The duals of the irreducible factors of $\mathbf{R}$ are themselves factors of $\mathbf{R}$.

Proof. Proposition 5.3 guarantees that every good partial resolution π' of W is very good. Hence, Proposition 4.11 implies that $\mathcal{W}^p_{\pi'}$ is the set of nonzero weights of a matter presentation $\mathbf{R}^p_{\pi'}$ for all closed points $p \in W_{\mathrm{sing}}$ and all $\pi' \in \mathrm{Good}(W)$. The representation $\mathbf{R}^p_{\pi'}$ is unique up to isomorphism and the additional and removal of trivial factors, as we showed in Section 4.2.

By Theorem 5.9(a) and Corollary 5.10, $\mathcal{W}^p_{\pi'}$ is independent of the choice of $\pi' \in \mathrm{Good}(W)$, and Part (a) follows. Part (b) follows from Theorem 5.9(b), and Part (c) follows from Propositions 4.10 and 4.11(a). To prove Part (d), note that $\mathfrak{R}^p = -\mathfrak{R}^p$ for all closed points $p \in W_{\mathrm{sing}}$. Hence, we have that $\mathcal{W}^p_\pi = -\mathcal{W}^p_\pi$—so that the representation $\mathbf{R}^p_\pi$ is self-dual—for all closed points $p \in W_{\mathrm{sing}}$. Part (d) follows. □

Corollary 5.13(a) implies that the local matter representations are independent of choice of a (very) good partial resolution when the gauge algebra $\mathfrak{g}$ is simply-laced and does not have a simple factor that is isomorphic to $\mathfrak{sl}_{n+1}$ for any $2 \leq n \leq 8$. In this case, Corollary 5.13(b) implies that KKMR decomposition is given by $\mathrm{I}(\mathfrak{g}, \mathbf{R})$, where $\mathfrak{g}$ is the gauge algebra and $\mathbf{R}$ is the sum of local matter representations. As the irreducible factors of local matter representations are (quasi-)minuscule, so are the irreducible factors of $\mathbf{R}$, as Corollary 5.13(c) guarantees. Because the local matter representations are self-dual, irreducible representations appear in $\mathbf{R}$ only in dual pairs, as Corollary 5.13(d) shows.

We can use Corollary 5.13 to obtain a list of possible KKMR decompositions for the case of a simple simply-laced gauge algebra that is not isomorphic to $\mathfrak{sl}_{n+1}$ for $2 \leq n \leq 8$, as we show explicitly in the following result.

Corollary 5.14. *In Situation 5.4, if W_{sing} has a unique positive-dimensional irreducible component (i.e., if $\mathfrak{R}$ is irreducible), then the following conclusions hold.*

(a) If $\mathfrak{R}$ is of type A_n for $n \geq 9$ and $|\operatorname{Good}(W)| > 1$, then the KKMR decomposition (1) *is one of* $\mathrm{I}(\mathfrak{sl}_{n+1}, \mathbf{vec})$, $\mathrm{I}(\mathfrak{sl}_n, \bigwedge^2 \mathbf{vec})$, *and* $\mathrm{I}(\mathfrak{sl}_{n+1}, \mathbf{vec} \oplus \bigwedge^2 \mathbf{vec})$, *where* $\mathbf{vec}$ *is the defining (first fundamental) representation of* $\mathfrak{sl}_{n+1}$.

(b) If $\mathfrak{R}$ is of type D_n for $n \geq 9$ and $|\operatorname{Good}(W)| > 1$, then the KKMR decomposition (1) *is* $\mathrm{I}(\mathfrak{so}_{2n}, \mathbf{vec})$, *where* $\mathbf{vec}$ *is the defining (vector) representation of* $\mathfrak{so}_{2n}$.

(c) If $\mathfrak{R}$ is of type D_n for $4 \leq n \leq 8$ and $|\operatorname{Good}(W)| > 1$, then the KKMR decomposition (1) *is one of* $\mathrm{I}(\mathfrak{so}_{2n}, \mathbf{vec})$, $\mathrm{I}(\mathfrak{so}_{2n}, \mathbf{spin}_+)$, *and* $\mathrm{I}(\mathfrak{so}_{2n}, \mathbf{vec} \oplus \mathbf{spin}_+)$, *where* $\mathbf{vec}$ *and* $\mathbf{spin}_+$ *are the defining (vector) representation and one of the half-spin representations of* $\mathfrak{so}_{2n}$*, respectively.*

(d) If $\mathfrak{R}$ is of type E_6 and $|\operatorname{Good}(W)| > 1$, then the KKMR decomposition (1) *is* $\mathrm{I}(\mathfrak{e}_6, \mathbf{27})$, *where* $\mathbf{27}$ *is one of the non-trivial minuscule representations of* $\mathfrak{e}_6$.

(e) If $\mathfrak{R}$ is of type E_7 and $|\operatorname{Good}(W)| > 1$, then the KKMR decomposition (1) *is* $\mathrm{I}(\mathfrak{e}_7, \mathbf{56})$, *where* $\mathbf{56}$ *is the non-trivial minuscule representation of* $\mathfrak{e}_7$.

(f) If $\mathfrak{R}$ is of type E_8, then we have that $|\operatorname{Good}(W)| = 1$.

Proof. The case of E_8 has already been dealt with in Corollary 5.7. We therefore only need to consider the other cases.

We first describe a general property of the decomposition $\mathrm{I}(\mathfrak{g}, \mathbf{R}')$, where $\mathfrak{g}$ is the gauge algebra. Let $\mathrm{Irrep}_{/*}(\mathfrak{g})$ denote the set of isomorphism classes of non-trivial non-adjoint finite-dimensional irreducible representations of the gauge algebra $\mathfrak{g}$ modulo duality. Given a locally finite representation $\mathbf{R}'$ of $\mathfrak{g}$, let $\mathrm{red}(\mathbf{R}') \subseteq \mathrm{Irrep}_{/*}(\mathfrak{g})$ denote the set of isomorphism classes of the non-trivial non-adjoint irreducible factors of $\mathbf{R}$ modulo duality. Because $\mathbf{w}$ and $-\mathbf{w}$ have the same normal hyperplanes for all nonzero weights $\mathbf{w}$ and the weights of the adjoint representation define the walls of the dual fundamental Weyl chamber, we have that

$$\text{the decomposition } \mathrm{I}(\mathfrak{g}, \mathbf{R}') \text{ depends on } \mathbf{R}' \text{ only through } \mathrm{red}(\mathbf{R}'). \tag{2}$$

Let $\mathbf{R}$ denote the representation defined in Corollary 5.13(b), which is a locally finite representation. By Corollary 5.13(c), any irreducible representation that appears in $\mathrm{red}(\mathbf{R})$ must be a minuscule representation whose highest weight has length at most $\sqrt{2}$. We now divide into cases to apply the classification of minuscule representations of simple Lie algebras to complete the proof of the corollary.

Case 1: $\mathfrak{R}$ is of type A_n for $n \geq 9$. The minuscule representations of $\mathfrak{sl}_{n+1}$ are fundamental representations $\bigwedge^i \mathbf{vec}$ for $1 \leq i \leq n$, where $\mathbf{vec}$ is the defining (first fundamental) representation. The ith and $(n+1-i)$th fundamental representations are dual. In light of the standard formulae for the inverses of the Cartan matrices of irreducible root systems [55, 83], the highest weight of the ith fundamental representation has length $\sqrt{\frac{i(n+1-i)}{n+1}}$. For $3 \leq i \leq n-2$ and $n \geq 9$, we have that

$$\frac{i(n+1-i)}{n} \geq \frac{3(n-2)}{n+1} = 3 - \frac{9}{n+1} > 2.$$

Hence, for $n \geq 9$, the class $\mathrm{red}(\mathbf{R})$ must be one of $\mathrm{red}(0)$, $\mathrm{red}(\mathbf{vec})$, $\mathrm{red}(\bigwedge^2 \mathbf{vec})$, and $\mathrm{red}(\mathbf{vec} \oplus \bigwedge^2 \mathbf{vec})$. Part (a) therefore follows from Corollary 5.10(b) and (2).

Case 2: $\mathfrak{R}$ is of type D_n. The minuscule representations of $\mathfrak{so}_{2n}$ are the defining (vector) representation $\mathbf{vec}$ and the half-spin representations $\mathbf{spin}_{\pm}$. As the half-spin representations are dual, $\mathrm{red}(\mathbf{R})$ must be one of $\mathrm{red}(0)$, $\mathrm{red}(\mathbf{vec})$, $\mathrm{red}(\mathbf{spin}_+)$, and $\mathrm{red}(\mathbf{vec} \oplus \mathbf{spin}_+)$. Part (c) therefore follows from Corollary 5.10(b) and(2).

To prove Part (b), note that the standard formulae for the inverses of the Cartan matrices of irreducible root systems [55, 83] imply that the highest weights of the half-spin representations have length $\sqrt{\frac{n}{4}}$, which is greater than $\sqrt{2}$ for $n \geq 9$. Hence, the only possibilities for red($\mathbf{R}$) when n is at least 9 are red(0) and red(**vec**). Part (b) therefore follows from Corollary 5.10(b) and (2).

Case 3: $\mathfrak{R}$ is of type E_6. The minuscule representations of $\mathfrak{e}_6$ are **27** and its dual. Hence, red($\mathbf{R}$) must be one of red(0) and red(**27**). Part (d) therefore follows from Corollary 5.10(b) and (2).

Case 4: $\mathfrak{R}$ is of type E_7. The unique minuscule representation of $\mathfrak{e}_7$ (up to isomorphism) is **56**. Hence, red($\mathbf{R}$) must be one of red(0) and red(**56**). Part (e) therefore follows from Corollary 5.10(b) and (2).

The cases clearly exhaust all possibilities, completing the proof of the corollary. □

6 Proofs of Propositions 4.10, 4.11, and 5.3 and Theorem 5.5

In this section, we prove all of results asserted in Sections 4 and 5 except for Theorem 5.9, which we prove in Section 7.

In Sections 6.1 and 6.2, we prove preliminary results. In particular, in Section 6.1, we relate the enhancement homomorphism to intersection numbers on good partial resolutions. In Section 6.2, we prove basic Lie-theoretic properties of roots and weights that have length at most $\sqrt{2}$.

In Sections 6.3–6.5, we complete arguments for the results asserted in Sections 4 and 5. Specifically, in Section 6.3, we prove basic properties of the enhancement weights and complete the proofs of Propositions 4.10 and 4.11. In Section 6.4, we characterize the movable cone and complete the proof of Proposition 5.3. In Section 6.5, we characterize the ample cone and complete the proof of Theorem 5.5.

6.1 Intersection theory on the partial resolution

In this section, we relate the enhancement homomorphism to intersection numbers on good partial resolutions. The first lemma relates the form $\langle -, - \rangle$ on $\mathfrak{L}$ to the degrees of line bundles on the one-dimensional schemes $\mathtt{C}_i^{(j)}$.

Lemma 6.1. *In Situation 4.1, let* $\mathbf{c} \in L$ *be such that* $\psi(\mathbf{c}) = c_1(\mathcal{L})$ *in* $\operatorname{Pic} Y_{/\mathrm{tors}}$. *We have that*

$$\deg_{\mathsf{C}_i^{(j)}} \mathcal{L} = -\langle \mathbf{c}_i^{(j)}, \mathbf{c} \rangle$$

for all $1 \leq i \leq s$ *and* $1 \leq j \leq k_i$.

Proof. By linearity, we can assume that $\mathbf{c} = n\mathbf{c}_{i'}^{(j')}$ for some $1 \leq i' \leq s$ and $1 \leq j' \leq k_{i'}$. If $i = i'$, then we have that

$$\deg_{\mathsf{C}_i^{(j)}} \mathcal{L} = -\langle \mathbf{c}_i^{(j)}, n\mathbf{c}_{i'}^{(j')} \rangle_{\mathcal{O}_{W,\eta_i}} = -\langle \mathbf{c}_i^{(j)}, n\mathbf{c}_{i'}^{(j')} \rangle$$

by the definitions of $\langle -, - \rangle_{\mathcal{O}_{W,\eta_i}}$ and $\langle -, - \rangle$. If $i \neq i'$, then as $\mathcal{L}$ is trivial in a neighborhood of Y_{η_i}, we have that

$$\deg_{\mathsf{C}_i^{(j)}} \mathcal{L} = 0 = -\langle \mathbf{c}_i^{(j)}, n\mathbf{c}_{i'}^{(j')} \rangle.$$

In either case, we have that

$$\deg_{\mathsf{C}_i^{(j)}} \mathcal{L} = -\langle \mathbf{c}_i^{(j)}, n\mathbf{c}_i^{(j')} \rangle,$$

as desired. □

The next two results provide alternative characterizations of the enhancement homomorphism, in terms of the fibers of the Cartan divisors and intersection numbers on the surfaces $\widetilde{Y}^p$, respectively. The proofs of these characterizations exploit the projection formula for intersection numbers [26, Proposition 2.5(c)].

Proposition 6.2. *In Situation 4.7, let* $i \in S_p$ *and* $1 \leq j \leq k_i$ *indices. For* $1 \leq k \leq k^p$, *let* $m_{i,k}^{j,p,\pi}$ *denote the multiplicity of* $(Z_i^{(j)})_p$ *along* $\pi_* C_k^p$. *We have that*

$$\left((\Lambda_\pi^p)_* \circ \theta_{\mathcal{O}_{H^p,p},\mathbb{Q}} \circ \phi_\pi^p\right)(\mathbf{c}_i^{(j)}) = \sum_{k=1}^{k^p} m_{i,k}^{j,p,\pi} [C_k^{p,\pi}],$$

as elements of $\operatorname{Cl}(Y \times_W \operatorname{Spec} \mathcal{O}_{H^p,p}) \otimes \mathbb{Q}$.

Proof. Let $n \in \mathbb{Z}_{>0}$ be such that $n\psi(\mathbf{c}_i^{(j)}) \in \operatorname{Pic} Y_{/\mathrm{tors}}$, which exists because Y is $\mathbb{Q}$-factorial. Let $\mathcal{L}$ be such that $c_1(\mathcal{L}) = n[Z_i^{(j)}]$ in $\operatorname{Pic} Y_{/\mathrm{tors}}$. By the definition of ϕ_π^p, we have that $\theta_{\mathcal{O}_{H^p,p},\mathbb{Q}} \circ \phi_\pi^p = (\Phi_\pi^p)_\mathbb{Q}^* \circ \psi$. Hence, we have that $(\theta_{\mathcal{O}_{H^p,p},\mathbb{Q}} \circ \phi_\pi^p)(n\mathbf{c}_i^{(j)}) =$

$(c_1 \circ (\Phi^p_\pi)^*)(\mathcal{L})$ in $\mathrm{Cl}(\widetilde{Y}^p) \otimes \mathbb{Q}$. As Λ^p_π is birational and $\Phi^p_\pi = \Gamma^p_\pi \circ \Lambda^p_\pi$, the projection formula [26, Proposition 2.5(c)] implies that

$$(\Lambda^p_\pi)_* \circ \theta_{\mathcal{O}_{H^p,p},\mathbb{Q}} \circ \phi^p_\pi)(n\mathbf{c}_i^{(j)}) = (c_1 \circ (\Gamma^p_\pi)^*)(\mathcal{L}).$$

Because $Y \times_W \operatorname{Spec} \mathcal{O}_{H^p,p}$ is normal (by Theorem 3.27), the definition of the first Chern class implies that

$$(c_1 \circ (\Gamma^p_\pi)^*)(\mathcal{L}) = n \sum_{k=1}^{k^p} m^{j,p,\pi}_{i,k} [C^{p,\pi}_k]$$

in $\mathrm{Cl}(Y \times_W \operatorname{Spec} \mathcal{O}_{H^p,p}) \otimes \mathbb{Q}$. Dividing by n yields the assertion of the proposition. □

Proposition 6.3. *In Situation 4.7, let* $\mathbf{c} \in \mathfrak{L}$ *be such that* $\psi(\mathbf{c}) = c_1(\mathcal{L})$ *in* $\operatorname{Pic} Y_{/\mathrm{tors}}$. *For all* $1 \leq k \leq k^p$, *we have that*

$$\deg_{\pi_* C^p_k} (\Gamma^p_\pi)^* \mathcal{L} = \deg_{C^p_k} (\Phi^p_\pi)^* \mathcal{L} = -\langle \mathbf{r}^p_k, \psi^p_\pi(\mathbf{c}) \rangle^p.$$

Proof. The first equality of the lemma follows from the projection formula [26, Proposition 2.5(c)]. To prove the second equality, note that $\theta_{\mathcal{O}_{H^p,p},\mathbb{Q}} \circ \phi^p_\pi = (\Phi^p_\pi)^*_{\mathbb{Q}} \circ \psi$ holds by the definition of ϕ^p_π. Hence, we have that $(\Phi^p_\pi)^* \mathcal{L} = (\theta_{\mathcal{O}_{H^p,p}} \circ \phi^p_\pi)(\mathbf{c})$. The second equality of the lemma therefore follows from the definition of $\langle -, - \rangle^p$. □

6.2 Lemmata on roots and weights of length at most $\sqrt{2}$

In this section, we prove properties of roots and weights of length at most $\sqrt{2}$. The first result shows that, for simply-laced root systems, every integral weight of length at most $\sqrt{2}$ is (quasi-)minuscule.

Lemma 6.4. *Let* $\mathfrak{R}$ *be a simply-laced root system. If weight* $\mathbf{w} \in \mathfrak{L}^\dagger(\mathfrak{R})$ *is such that* $\langle \mathbf{w}, \mathbf{w} \rangle \leq 2$, *then* $\mathbf{w}$ *is (quasi-)minuscule.*

Proof. It suffices to prove the result for irreducible root systems. We divide into cases based on the type of the root system to complete the argument.

Case 1: $\mathfrak{R}$ is of type A_n. We claim that the only non-fundamental dominant weight of length at most $\sqrt{2}$ is the highest root. This claim can be verified straightforwardly for $n = 1, 2$. We next consider the case of $n \geq 3$. Let $\mathbf{w}_i$ denote the ith fundamental weight. The standard formulae for the inverses of the Cartan matrices of irreducible root systems [55, 83] imply that $\langle \mathbf{w}_i, \mathbf{w}_j \rangle = \min\{i, j\} - \frac{ij}{n+1}$. It follows that, for all $i \leq j$ with $i + j \leq n + 1$, we have

$$\begin{aligned}\langle \mathbf{w}_i + \mathbf{w}_j, \mathbf{w}_i + \mathbf{w}_j \rangle &= 4i + 2j - \frac{i^2 + 2ij + j^2}{n+1} \\ &= 2i + \frac{2(i+j)(n+1) - (i+j)^2}{n+1} \geq i \geq 1,\end{aligned}$$

with equality if and only if $i = 1$ and $j = n$. Analogous logic applies for $i+j \geq n+1$. Hence, the only sum of two fundamental weights that has length at most $\sqrt{2}$ is $\mathbf{w}_1 + \mathbf{w}_n$, which is the highest root. The claim follows.

Because the fundamental representations of $\mathfrak{sl}_{n+1}$ are all minuscule, the previous paragraph implies that the lemma holds in the case of type A.

Case 2: $\mathfrak{R}$ is of type D_n for $n \geq 4$. The standard formulae for the inverses of the Cartan matrices of irreducible root systems [55, 83] imply that the lengths of all fundamental weights are all greater than 1. Hence, all dominant minuscule weights of length at most $\sqrt{2}$ are fundamental. For $n \leq 8$, the only fundamental weights of length at most $\sqrt{2}$ are the first two and the last two, which are the highest weight of the defining (vector) representation of $\mathfrak{so}_{2n}$, the highest root, and the highest weights of the two half-spin representations of $\mathfrak{so}_{2n}$, respectively. For $n \geq 9$, the only fundamental weights of length at most $\sqrt{2}$ are the first and second, which are the highest weight of the defining (vector) representation of $\mathfrak{so}_{2n}$ and the highest root, respectively. Because the defining representation and the half-spin representations are minuscule, the lemma holds in either case.

Case 3: $\mathfrak{R}$ is of type E_6, E_7, or E_8. The standard formulae for the inverses of the Cartan matrices of irreducible root systems [55, 83] imply that the lengths of all fundamental weights are all greater than 1. Hence, all dominant minuscule weights of length at most $\sqrt{2}$ are fundamental. In the case of E_6, the fundamental weights of length at most $\sqrt{2}$ are the first, fifth, and sixth, the first of which is the highest root and the last two of which are the highest weights of (27-dimensional) minuscule representations. In the case of E_7, the fundamental weights of length at most $\sqrt{2}$ are the first and sixth, which are the highest weights of the 56-dimensional minuscule representations and the highest root, respectively. In the case of E_8, the only fundamental weight of length at most $\sqrt{2}$ is the highest root. In all cases, the lemma holds.

Theorem 2.2 implies that the cases exhaust all possibilities, completing the proof of the lemma. □

The second result shows that short roots span the ambient vector spaces of irreducible root systems.

Proposition 6.5. *If $\mathfrak{R}$ is an irreducible root system in a vector space V, then the set of roots of length $\sqrt{2}$ span V.*

Proof. Let α be any root with $\langle \alpha, \alpha \rangle = 2$. Because $\mathfrak{W}(\mathfrak{R})$ acts via orthogonal transformations and preserves $\mathfrak{R}$, the set $\mathfrak{W}(\mathfrak{R})\alpha$ consists entirely of roots of length $\sqrt{2}$. By the irreducibility of the action of $\mathfrak{W}(\mathfrak{R})$ on V for any irreducible root system $\mathfrak{R}$ [6, Chapitre V, §4.7, Proposition 7], the set $\mathfrak{W}(\mathfrak{R})\alpha$ must span V. □

The third and final result provides an upper bound on the number of weights of length at most $\sqrt{2}$.

Lemma 6.6. *If $\mathfrak{R}$ is a root system in an inner product space E, then $\mathfrak{L}^\dagger(\mathfrak{R})$ has at most $2^{2\dim E} - 1$ elements of with length at most $\sqrt{2}$ that are not proportional to roots.*

Proof. Let $\mathfrak{R}^s$ denote the set of elements of $\mathfrak{R}$ that have length $\sqrt{2}$. It follows from Proposition 6.5 that $\mathfrak{R}^s$ is a root system in E. Note that $\mathfrak{L}^\dagger(\mathfrak{R}) \subseteq \mathfrak{L}^\dagger(\mathfrak{R}^s)$. Hence, we can assume that $\mathfrak{R}$ is simply-laced, so that $\mathfrak{R} = \mathfrak{R}^s$. As $\mathfrak{L}^\dagger(\mathfrak{R}_1 \oplus \mathfrak{R}_2) = \mathfrak{L}^\dagger(\mathfrak{R}_1) \oplus \mathfrak{L}^\dagger(\mathfrak{R}_2)$, we can further assume that $\mathfrak{R}$ is irreducible. By Lemma 6.4, it suffices to show that there are at most $2^{2\dim E} - 1$ minuscule weights. By Theorem 2.2, $\mathfrak{R}$ must be of one of types A, D, and E, and we divide into cases to complete the proof.

Case 1: $\mathfrak{R}$ is of type A_n. The minuscule representations of $\mathfrak{sl}_{n+1}$ are $\bigwedge^i \mathbf{vec}$ for $1 \le i \le n$, where **vec** is the defining (first fundamental) representation. In particular, there are at most

$$|\,\text{Q-Minusc}(\mathfrak{R})| \le \sum_{i=1}^{n} \binom{n+1}{i} = 2^{n+1} - 2 < 2^{2n} - 1,$$

as desired.

Case 2: $\mathfrak{R}$ is of type D_n for $n \geq 4$. The minuscule representations of $\mathfrak{so}_{2n}$ are the defining (vector) representation **vec** and the half-spin representations $\mathbf{spin}_+$ and $\mathbf{spin}_-$. These representations have dimensions $2n$, 2^{n-1}, and 2^{n-1}, respectively. Hence, we have that

$$|\,\text{Q-Minusc}(\mathfrak{R})| \leq 2n + 2^{n-1} + 2^{n-1} \leq 2^n + 2^n = 2^{n+1} < 2^{2n},$$

as desired.

Case 3: $\mathfrak{R}$ is of type E_6, E_7, or E_8. In the case of E_6, there are two non-isomorphic 27-dimensional minuscule representations and no other minuscule representations. Hence, there are $54 < 2^{12}$ minuscule weights. In the case of E_7, there is a unique (up to isomorphism) minuscule representation, which is 56-dimensional. Hence, there are $56 < 2^{14}$ minuscule weights. In the case of E_8, there is no minuscule representation.

We have exhausted all possible cases and have proven that there are at most $2^{2\dim E}-1$ minuscule weights in all cases—completing the proof of the lemma. □

6.3 Basic properties of the enhancement weights

We now prove the basic properties of the enhancement homomorphism in Situation 4.7. The first result shows that ϕ_π^p restricts to 0 on $\mathfrak{L}_i$ for all $i \notin S_p$.

Lemma 6.7. *In Situation 4.7, we have that $\phi_\pi^p|_{\mathfrak{L}_i} = 0$ for all $i \notin S_p$, and hence that $\mathcal{W}_\pi^p \subseteq \mathfrak{L}_p \otimes \mathbb{R}$.*

Proof. To prove the first assertion, it suffices to show that $\phi_\pi^p(\mathbf{c}_i^{(j)}) = 0$ for all $i \in S_p$ and $1 \leq j \leq k_i$. Let $i \in S_p$ and $1 \leq j \leq k_i$. Let $n \in \mathbb{Z}_{>0}$ be such that $n\psi(\mathbf{c}_i^{(j)}) \in \operatorname{Pic} Y_{/\text{tors}}$, which exists because Y is $\mathbb{Q}$-factorial. Let $\mathcal{L}$ be such that $c_1(\mathcal{L}) = n[Z_i^{(j)}]$ in $\operatorname{Pic} Y_{/\text{tors}}$. As $p \notin Z_i$, the invertible sheaf $\mathcal{L}$ is trivial in an open neighborhood of Y_p. Hence, the class of $(\Gamma_\pi^p)^*\mathcal{L}$ in $\operatorname{Pic} Y \times_W \operatorname{Spec} \mathcal{O}_{H^p,p}$ is trivial. As a result, we have that $(\Gamma_\pi^p)^*_{\mathbb{Q}}[Z_i] = 0$ and hence that $\phi_\pi^p(\mathbf{c}_i^{(j)}) = 0$. Because i and j were arbitrary, we have proven the first assertion. The second assertion follows from the first. □

The second result shows that ϕ_π^p is an isometry if all of irreducible components of W_{sing} that pass through p are nonsingular at p.

Lemma 6.8. *In Situation 4.7, if Z_i is nonsingular at p for all $i \in S_p$, then we have that $\langle \phi_\pi^p(-), \phi_\pi^p(-)\rangle^p = \langle -, - \rangle$ as forms on $\mathfrak{L}_p$. Hence, we have that $\langle \mathbf{w}, \mathbf{w}\rangle \leq 2$ for all $w \in \mathcal{W}_\pi^p$.*

The proof of Lemma 6.8 exploits the results of Section 6.1 and the constancy of the degrees of invertible sheaves in flat families [81, Corollary 24.7.3].

Proof. For all $i \in S_p$ and $1 \leq j \leq k$, the scheme $Z_i^{(j)}$ is integral and dominates Z_i, which is nonsingular at p. Hence, $Z_i^{(j)}$ must be flat over Z_i at p by the standard criterion for flatness over discrete valuation rings [31, Proposition III.9.7].

Let $i' \in S_p$ and $1 \leq j' \leq k_{i'}$ be arbitrary indices. Let $n \in \mathbb{Z}_{>0}$ be such that $n\psi(\mathbf{c}_{i'}^{(j')}) \in c_1(\operatorname{Pic} Y_{/\text{tors}})$, which exists because Y is $\mathbb{Q}$-factorial. Let $\mathcal{L}$ be such that $c_1(\mathcal{L}) = n[Z_i^{(j)}]$ in $\operatorname{Pic} Y_{/\text{tors}}$. Due to the constancy of the degrees in flat families [81, Corollary 24.7.3], we have that

$$\deg_{(Z_i^{(j)})_p} \mathcal{L} = \deg_{\mathtt{C}_i^{(j)}} \mathcal{L}.$$

By Lemma 6.1, it follows that

$$\deg_{(Z_i^{(j)})_p} \mathcal{L} = \deg_{\mathtt{C}_i^{(j)}} \mathcal{L} = -\langle \mathbf{c}_i^{(j)}, n\mathbf{c}_{i'}^{(j')}\rangle.$$

Proposition 6.2 and the projection formula [26, Proposition 2.5(c)] imply that

$$\deg_{(\theta_{H^p,p,\mathbb{Q}} \circ \phi_\pi^p)(\mathbf{c}_i^{(j)})} (\Phi_\pi^p)^* \mathcal{L} = \deg_{(Z_i^{(j)})_p} \mathcal{L} = -\langle \mathbf{c}_i^{(j)}, n\mathbf{c}_{i'}^{(j')}\rangle.$$

In light of Proposition 6.3, it follows that

$$-\langle \phi_\pi^p(\mathbf{c}_i^{(j)}), n\phi_\pi^p(\mathbf{c}_{i'}^{(j')})\rangle^p = \deg_{(\theta_{H^p,p,\mathbb{Q}} \circ \phi_\pi^p)(\mathbf{c}_i^{(j)})} (\Phi_\pi^p)^* \mathcal{L} = -\langle \mathbf{c}_i^{(j)}, n\mathbf{c}_{i'}^{(j')}\rangle.$$

As i, j, i', and j' were arbitrary, we have proven the first assertion of the proposition, and the second assertion follows from the first. □

The third result shows that the image of ϕ_π^p lies in $\mathfrak{L}^p$ (instead of merely $\mathfrak{L}^p \otimes \mathbb{Q}$) if all of the Cartan divisors are Cartier.

Lemma 6.9. *In Situation 4.7, if $[Z_i^{(j)}]$ is Cartier for all $i \in S_p$ and $1 \leq j \leq k_i$, then the image of ϕ_π^p lies in $\mathfrak{L}^p \subseteq \mathfrak{L}^p \otimes \mathbb{Q}$. In this case, we have that $\mathcal{W}_\pi^p \subseteq L_p^\dagger$.*

Proof. The hypothesis of the lemma implies that the image of ψ lies in $\operatorname{Pic} Y_{/\mathrm{tors}} \subseteq \operatorname{Pic} Y \otimes \mathbb{Q}$. It therefore follows from Proposition 6.2 that the image of $\theta_{\mathcal{O}_{H^p},p,\mathbb{Q}} \circ \phi_\pi^p$ lies in the set of integral divisors that are supported on the exceptional locus of $\widetilde{\pi}^p$. Applying $\theta_{\mathcal{O}_{H^p},p,\mathbb{Q}}^{-1}$ yields the first assertion of the lemma, and the second assertion is direct subsequent the first assertion. □

We are now ready to prove Propositions 4.10 and 4.11.

Proof of Propositions 4.10 and 4.11. Proposition 4.10 follows from Lemmata 6.7, 6.8, and 6.9. Proposition 4.11(a) follows from Proposition 4.10 and Lemma 6.4.

It remains to prove Proposition 4.11(b). As reflections over the hyperplanes that are normal to roots generate the Weyl group, it suffices to prove that $s_{\mathbf{c}}\mathcal{W}_\pi^p = \mathcal{W}_\pi^p$ for all $\mathbf{c} \in \mathfrak{R}$. Let $\mathbf{c} \in \mathfrak{R}$ be arbitrary. It follows from Proposition 4.10 that $s_{\phi_\pi^p(\mathbf{c})}\mathbf{c} \circ \phi_\pi^p = \phi_\pi^p \circ s_{\mathbf{c}}$. As reflections are self-adjoint, we have that $s_{\mathbf{c}} \circ (\phi_\pi^p)^\dagger = (\phi_\pi^p)^\dagger \circ s_{\phi_\pi^p(\mathbf{c})}$. Propositions 2.4 and 4.10 imply that $\phi_\pi^p(\mathbf{c}) \in \mathfrak{R}^p$. Hence, $s_{\phi_\pi^p(\mathbf{c})}$ defines a bijection from $\mathfrak{R}^p$ to $\mathfrak{R}^p$. It follows that

$$s_{\mathbf{c}}(\mathcal{W}_\pi^p) = (s_{\mathbf{c}} \circ (\phi_\pi^p)^\dagger)(\mathfrak{R}^p) = ((\phi_\pi^p)^\dagger \circ s_{\phi_\pi^p(\mathbf{c})})(\mathfrak{R}^p) = (\phi_\pi^p)^\dagger(\mathfrak{R}^p) = \mathcal{W}_\pi^p$$

as multisets, as desired. □

6.4 The movable cone

To study the movable cone, we first relate $\mathfrak{L}$ to the relative Picard group.

Proposition 6.10. *In Situation 4.1, if W is $\mathbb{Q}$-factorial and $\pi : Y \to W$ is a good partial resolution, then $\psi_{\mathbb{R}} = \psi \otimes \mathbb{R}$ induces an isomorphism $\mathfrak{L} \otimes \mathbb{R}$ to $N^1(Y/W)$.*

The proof of Proposition 6.10 relies on Lemma 6.1. We use two results on flatness—generic flatness [29, Corollaire 6.9.3] and the constancy of the degrees of invertible sheaves in flat families [81, Corollary 24.7.3]—to relate the degree expression that appears in the statement of Lemma 6.1 to the degree of an invertible sheaf along a curve.

Proof. We first verify that $\psi_{\mathbb{R}}$ is injective. Suppose for sake of deriving a contradiction that there exists

$$\mathbf{c} = \sum_{i=1}^{s} \sum_{j=1}^{k_i} \gamma_i^{(j)} \mathbf{c}_i^{(j)}$$

with $\psi(\mathbf{c})$ numerically π-trivial, where the coefficients $\gamma_i^{(j)}$ are integral and some coefficient is nonzero. As $\langle -, - \rangle$ is nondegenerate, there exist $1 \leq i \leq s$ and $1 \leq j \leq k_i$ such that $\langle \mathbf{c}, \mathbf{c}_i^{(j)} \rangle \neq 0$. Let $n \in \mathbb{Z}_{>0}$ and $\mathcal{L} \in \operatorname{Pic} Y$ be such that $c_1(\mathcal{L}) = n\psi(D)$—which exist because Y is $\mathbb{Q}$-factorial. By Lemma 6.1, we have that

$$\deg_{\mathsf{C}_i^{(j)}} \mathcal{L} \neq 0.$$

By generic flatness [29, Corollaire 6.9.3], there exists a closed point p such that $Z_i^{(j)}$ is flat over Z_i at p. Due to the constancy of the degrees of invertible sheaves in flat families [81, Corollary 24.7.3], we have that

$$\deg_{(Z_i^{(j)})_p} \mathcal{L} = \deg_{\mathsf{C}_i^{(j)}} \mathcal{L} \neq 0.$$

Hence, $\mathcal{L}$ is not numerically π-trivial, and it follows that $\psi(\mathbf{c})$ is not numerically π-trivial either. Because $\mathbf{c}$ was arbitrary, we have proven that $\psi_{\mathbb{R}}$ is injective.

We next verify that $\psi_{\mathbb{R}}$ is surjective. Let $\mathcal{L} \in \operatorname{Pic} Y$ be arbitrary, and let D be a divisor that represents $\mathcal{L}$. Let $n \in \mathbb{Z}_{>0}$ be such that $n\pi_* D$ is Cartier. Suppose that $n\pi_* D$ has class $c_1(\mathcal{L}_2)$. Note that $\mathcal{L}^{\otimes n}$ is numerically equivalent to $\mathcal{L}^{\otimes n} \otimes \mathcal{L}_2^{-1}$ over W. Furthermore, the divisor $nD - \pi^*(n\pi_* D)$ represents $\mathcal{L}^{\otimes n} \otimes \mathcal{L}_2^{-1}$. By construction, the divisor $nD - \pi^*(n\pi_* D)$ is supported on the exceptional locus of π, and hence can be expressed as a $\mathbb{Z}$-linear combination of the classes $[Z_i^{(j)}]$ for $1 \leq i \leq s$ and $1 \leq j \leq k_i$. As a result, $nc_1(\mathcal{L})$ is in the image of ψ. Because $\mathcal{L}$ was arbitrary, we have proven that $\psi_{\mathbb{R}}$ is surjective. □

In light of Proposition 6.10, we can identify $N^1(Y/W)$ and $N_1(Y/W)$ via $\psi_{\mathbb{R}}$ and $\langle -, - \rangle$ in Situation 4.1 when W is $\mathbb{Q}$-factorial.

The second result characterizes the movable cone.

Lemma 6.11. *In Situation 4.1, let $\pi : Y \to W$ is a good partial resolution and let $\mathbf{c} \in \mathfrak{L} \otimes \mathbb{R}$. We have that $\psi_{\mathbb{R}}(\mathbf{c}) \in \operatorname{Mov}(Y/W)$ if and only if $\langle \mathbf{c}, \mathbf{c}_i^{(j)} \rangle \leq 0$ holds for all $1 \leq i \leq s$ and $1 \leq j \leq k_i$.*

The proof of Lemma 6.11 relies on the Lipman [54] intersection-theoretic characterization of globally generated invertible sheaves on resolutions of local rings with rational singularities (Proposition 3.23).

Proof. We first prove the “only if” direction of the first assertion. Suppose that $\psi_{\mathbb{R}}(\mathbf{c}) \in \operatorname{Mov}(Y/W)$, so that $\psi_{\mathbb{R}}(\mathbf{c}) = \sum_{k=1}^{n} \gamma_k c_1(\mathcal{L}_k)$ holds in $N^1(Y/W)$, where $\gamma_k > 0$ and $\mathcal{L}_k$ is π-movable for all k. Let $1 \le i \le s$ be an arbitrary index. By the definition of movability, the codimension of the cokernel of $\pi^*\pi_*\mathcal{L}_k \to \mathcal{L}_k$ in Y is at least 2. It follows that the relative complete linear series $(\mathcal{L}_k, (\widehat{\pi}_i)_*\mathcal{L}_k)$ has only isolated points as base points, where we define $\widehat{\pi}_i = \pi \times_W \operatorname{Spec} \mathcal{O}_{W,\eta_i} : Y \times_W \operatorname{Spec} \mathcal{O}_{W,\eta_i} \to \operatorname{Spec} \mathcal{O}_{W,\eta_i}$. Therefore, $\mathcal{L}_k$ is $\widehat{\pi}$-nef. By Lemma 6.1, it follows that $\langle \mathbf{c}, \mathbf{c}_i^{(j)} \rangle \le 0$ for all $1 \le i \le s$ and $1 \le j \le k_i$.

We next prove the “if” direction of the first assertion. It suffices to prove that if $\mathbf{c} \in \mathfrak{L}$ is such that $\psi(\mathbf{c}) = c_1(\mathcal{L})$ and $\langle \mathbf{c}, \mathbf{c}_i^{(j)} \rangle < 0$ holds for all $1 \le i \le s$ and $1 \le j \le k_i$, then $\mathcal{L}$ is movable. Consider $\mathbf{c}$ and $\mathcal{L}$ as in the preceding sentence. Lemma 6.1 implies that

$$\deg_{\mathsf{C}_i^{(j)}} \mathcal{L} \ge 0$$

for all $1 \le i \le s$ and $1 \le j \le k_i$.

Fix an index $1 \le i \le s$. Proposition 4.4(a) implies that $\mathcal{O}_{W,\eta_i}$ has a Du Val singularity, hence in particular rational singularities. By Proposition 3.23, the natural homomorphism $(\widehat{\pi}_i)_*(\widehat{\pi}_i)^*\mathcal{L} \to \mathcal{L}$ is surjective, where $\widehat{\pi}_i = \pi \times_W \operatorname{Spec} \mathcal{O}_{W,\eta_i} : Y \times_W \operatorname{Spec} \mathcal{O}_{W,\eta_i} \to \operatorname{Spec} \mathcal{O}_{W,\eta_i}$. Hence, the support of the cokernel of the natural homomorphism $\pi^*\pi_*\mathcal{L} \to \mathcal{L}$ does not contain $\eta_i^{(j)}$ for any $1 \le j \le k_i$.

By Theorem 3.27, the fibers of π all have dimension at most 1. Hence, the points $\eta_i^{(j)}$ are the only codimension 1 points of Y that lie in the exceptional locus of π. Because i was arbitrary, it follows that the support of the cokernel of the natural homomorphism $\pi^*\pi_*\mathcal{L} \to \mathcal{L}$ has codimension at least 2 in Y, as desired. □

We are now ready to prove Proposition 5.3.

Proof of Proposition 5.3. By Theorem 3.40, it suffices to show that if $\pi : Y \to W$ is a very good partial resolution and $\pi' : Y' \to W$ is related to π by an extremal flop, then π' is a very good partial resolution. Let $\xi : Y \to X$ be an extremal flopping contraction over W with flop is $\xi' : Y' \to X$. If a closed point $p \in W$ satisfies Condition (i) for π, then Theorem 3.36(b) implies that p satisfies Condition (i) for π'.

Let $p \in W$ be a closed point that satisfies Condition (ii) for π. Proposition 6.10 guarantees that $\psi_{\mathbb{R}}^{-1}$ is well-defined. To prove that p satisfies Condition (ii) for π', it suffices to show that ξ does not contract any curve C_k^p. Suppose for sake of deriving a contradiction that ξ contracts curve C_k^p. We derive a contradiction by showing that no divisor that is ample on X can fail to be ξ-ample.

By Condition (ii) and Proposition 6.10, the class of C_k^p in $N_1(Y/W)$ is proportional to $\psi_{\mathbb{R}}(\mathbf{c})$ for some root $\mathbf{c} \in \mathfrak{R}$. Without loss of generality, we can assume that the class of C_k^p in $N_1(Y/W)$ is a positive multiple of $\psi_{\mathbb{R}}(\mathbf{c})$. Let $D' \in \mathrm{Amp}(Y'/X)$ be arbitrary, so that ξ is a D'-flopping contraction and ξ' is the D'-flop of ξ. As $-D'$ is ξ-ample, Theorem 3.3 and Proposition 6.3 imply that $\langle -\psi_{\mathbb{R}}^{-1}(D'), \mathbf{c}\rangle < 0$, so that $\langle \psi_{\mathbb{R}}^{-1}(D'), \mathbf{c}\rangle > 0$. As $D' \in \overline{\mathrm{Mov}}(Y/X)$ by Theorem 3.37, Lemma 6.11 implies that $\mathbf{c}$ cannot be a positive linear combination of simple roots, so that $\mathbf{c}$ must be a negative root (for any polarization for which the simple roots are the $\mathbf{c}_i^{(j)}$).

Now let $D \in \mathrm{Amp}(Y/X)$ be arbitrary. By Theorem 3.3 and Proposition 6.3, we must have that $\langle \psi_{\mathbb{R}}^{-1}(D), \mathbf{c}\rangle < 0$. Because $\mathbf{c}$ is a negative root, Lemma 6.11 implies that $D \notin \overline{\mathrm{Mov}}(Y/X)$, contradicting the fact that ample divisors are movable (see, e.g., Theorem 3.37). Hence, we can conclude that ξ cannot contract any curve C_k^p, and the proposition follows. □

6.5 The cone of curves and the ample cone

We now relate the enhancement weights to the cone of curves and the ample cone. The first result characterizes the cone of curves.

Proposition 6.12. *In Situation 4.1, if W is $\mathbb{Q}$-factorial and $\pi : Y \to W$ is a good partial resolution, then $\psi_{\mathbb{R}}^{-1}(\mathrm{NE}(Y/W))$ is spanned by $\bigcup_{p\in W_{\mathrm{sing}}(\mathbb{C})} \mathcal{W}_\pi^{p,+}$.*

Proof. Let C be a curve in Y that maps to a closed point $p \in W$. As π is a partial resolution, we must have that $p \in W_{\mathrm{sing}}$. Hence, we have that $C = \pi_* C_k^p$ for some $1 \leq k \leq k^p$. By Proposition 6.3 and Proposition 6.10, the class of C in $N_1(Y/W)$ is $(\phi_\pi^p)^\dagger(\mathbf{r}_k^p)$. We have therefore shown that $\mathrm{NE}(Y/W)$ lies in the span of $\bigcup_{p\in W_{\mathrm{sing}}(\mathbb{C})} \mathcal{W}_\pi^{p,+}$.

Conversely, we need to show that $\mathbf{w} \in \mathrm{NE}(Y/W)$ for all closed points $p \in W_{\mathrm{sing}}$ and all $\mathbf{w} \in \mathcal{W}_\pi^{p,+}$. By Proposition 2.1 and the linearity of $(\phi_\pi^p)^\dagger$, it suffices to show that $(\phi_\pi^p)^\dagger(\mathbf{r}_k^p) \in \mathrm{NE}(Y/W)$. If $\pi_*\mathbf{r}_k^p = 0$, then we have that $(\phi_\pi^p)^\dagger(\mathbf{r}_k^p) = 0$ by Proposition 6.3. If $\pi_*\mathbf{r}_k^p \neq 0$, then $(\phi_\pi^p)^\dagger(\mathbf{r}_k^p)$ is the class of $\pi_* C_k^p$ in $N_1(Y/W)$ by Proposition 6.3 and Proposition 6.10. In either case, we have that $(\phi_\pi^p)^\dagger(\mathbf{r}_k^p) \in \mathrm{NE}(Y/W)$, as desired. □

The second result applies our combinatorial results on the enhancement weights from Section 6.3 to show that the cone of curves is closed.

Corollary 6.13. *In Situation 5.4, we have that* $\psi_{\mathbb{R}}^{-1}(\mathrm{NE}(Y/W)) = \psi_{\mathbb{R}}^{-1}(\overline{\mathrm{NE}}(Y/W))$ *and* $\bigcup_{p \in W_{\mathrm{sing}}(\mathbb{C})} \mathcal{W}_\pi^{p,+}$ *spans both cones.*

Proof. By Proposition 6.12, the cone $\psi_{\mathbb{R}}^{-1}(\mathrm{NE}(Y/W))$ is spanned by $\bigcup_{p \in W_{\mathrm{sing}}(\mathbb{C})} \mathcal{W}_\pi^{p,+}$. Proposition 4.10 guarantees that every element of $\bigcup_{p \in W_{\mathrm{sing}}(\mathbb{C})} \mathcal{W}_\pi^{p,+} \subseteq \text{Q-Minusc}(\mathfrak{R})$ has length at most $\sqrt{2}$. But $\mathfrak{L}^\dagger(\mathfrak{R})$ can only contain finitely many elements of length at most $\sqrt{2}$. As a result, $\psi_{\mathbb{R}}^{-1}(\mathrm{NE}(Y/W))$ is spanned by a finite set, and is hence a closed cone. It follows that $\psi_{\mathbb{R}}^{-1}(\overline{\mathrm{NE}}(Y/W)) = \psi_{\mathbb{R}}^{-1}(\mathrm{NE}(Y/W))$ □

We are now ready to complete the proof of Theorem 5.5.

Proof of Theorem 5.5. Part (a) is the special case of Proposition 6.10 and Lemma 6.11 in which we are in Situation 5.4. Part (b) can be proven directly by combining Theorem 3.3, Corollary 6.13, and Part (a).

To prove Part (c), note that Part (a) implies that $\mathrm{Mov}(Y/W)$ is closed. Theorem 3.37 hence implies that there is a locally polyhederal KKMR decomposition

$$\psi_{\mathbb{R}}^{-1}(\mathrm{Mov}(Y/W)) = \text{closure of} \bigcup_{(\pi' : Y' \to W) \in \mathrm{Good}(W)} \psi_{\mathbb{R}}^{-1}(\overline{\mathrm{Amp}}(Y'/W)) \text{ in } \mathfrak{L} \otimes \mathbb{R},$$

where the cones $\psi_{\mathbb{R}}^{-1}(\mathrm{Amp}(Y'/W))$ are pairwise disjoint. There is also a finite, polyhedral decomposition of $\overline{\mathrm{Mov}}(Y/W)$ into closed chambers for the hyperplane arrangement consisting of hyperplanes that are orthogonal to weights of length at most $\sqrt{2}$— a decomposition that we call the *short-weight hyperplane decomposition*. Note that there only finitely many weights of length at most $\sqrt{2}$. Part (b) implies that the short-weight hyperplane decomposition refines the KKMR decomposition of $\mathrm{Mov}(Y/W)$, and Part (c) follows.

To prove Part (d), we continue to use the terminology of the previous paragraph. In light of Lemma 6.6, there are at most $2^{-1+\sum_{i=1}^s k_i}$ hyperplanes involved in the short-weight hyperplane decomposition. Hence, the short-weight hyperplane decomposition contains at most $2^{2^{-1+\sum_{i=1}^s k_i}}$ chambers. As the short-weight hyperplane decomposition refines the KKMR decomposition, the KKMR decomposition must contain at most $2^{2^{-1+\sum_{i=1}^s k_i}}$ chambers as well, and Part (d) follows. □

7 Proof of Theorem 5.9

Our primary tool to show that two enhancement homomorphisms give rise to the same multisets of enhancement weights is to show that the homomorphisms are conjugate under the action of the Weyl group—a property that we call Weyl-relatedness.

Definition 7.1. In Situation 4.1, for closed points $p \in W_{\mathrm{sing}}$ and $\pi, \pi' \in \mathrm{Good}(W)$, we say that ϕ^p_π and $\phi^p_{\pi'}$ are *Weyl-related* if there exists $w \in \mathfrak{W}(\mathfrak{R}^p)$ such that $\phi^p_{\pi'} = w \circ \phi^p_\pi$.

Lemma 7.2. *In Situation 4.1, if p is a closed point of W_{sing} and $\pi, \pi' \in \mathrm{Good}(W)$ are such that ϕ^p_π and $\phi^p_{\pi'}$ are Weyl-related, then we have that $\mathcal{W}^p_\pi = \mathcal{W}^p_{\pi'}$ as multisets.*

Proof. Suppose that $w \in \mathfrak{W}(\mathfrak{R}^p)$ is such that $\phi^p_{\pi'} = w \circ \phi^p_\pi$. Because w is an orthogonal operator, we have that $w^\dagger = w^{-1}$ and hence that $(\phi^p_{\pi'})^\dagger \circ w = (\phi^p_\pi)^\dagger$. Recall that $\mathcal{W}^p_\pi = (\phi^p_\pi)^\dagger(\mathfrak{R}^p)$ and $\mathcal{W}^p_{\pi'} = (\phi^p_{\pi'})^\dagger(\mathfrak{R}^p)$ by definition. As w induces a bijection from $\mathfrak{R}^p$ to itself, we have that

$$\mathcal{W}^p_\pi = (\phi^p_\pi)^\dagger(\mathfrak{R}^p) = (\phi^p_{\pi'})^\dagger(w(\mathfrak{R}^p)) = (\phi^p_{\pi'})^\dagger(\mathfrak{R}^p) = \mathcal{W}^p_{\pi'}$$

as multiset. The lemma follows. □

The key to the proof of Theorem 5.9(a) is the following result, which exploits the condition on the lengths of weights to show that very good partial resolutions that are related by extremal flops give rise to Weyl-related enhancement homomorphisms.

Lemma 7.3. *In Situation 5.4, let $p \in W_{\mathrm{sing}}$ be a closed point such that Y is non-singular above p. Let $\zeta : Y \to X$ be an extremal flopping contraction over W, let $\zeta' : Y' \to W$ be the flop of ζ, and let π' denote the composite $Y' \to X \to W$. Suppose that ζ contracts curve C^p_k and let $\mathbf{w} = (\phi^p_\pi)^\dagger(\mathbf{r}^p_k)$ denote the corresponding weight. If we have that $\langle \mathbf{w}, \mathbf{w} \rangle > \frac{8}{9}$, then ϕ^p_π and $\phi^p_{\pi'}$ are Weyl-related.*

To prove Lemma 7.3, we consider a short root $\mathbf{c} \in \mathfrak{R}$ whose associated divisor is not Cartier on the contracted variety X. We consider the difference between the images of $\mathbf{c}$ under the enhancement homorphisms for the two very good partial resolutions. We divide into cases based on the length of the difference to construct elements of the Weyl group that relate the enhancement homomorphisms. The bound on the length of the weight corresponding to the flopping curve is crucial to constraining the length of the difference and its inner products with other roots.

Proof. By passing to an open neighborhood of p, we can assume that $S_p = \{1, \ldots, s\}$. (This operation does not affect $\phi^p_\pi|_{\mathfrak{L}_p}$ or $\phi^p_{\pi'}|_{\mathfrak{L}_p}$ due to Proposition 4.10.) In this case, we have that $\mathfrak{L} = \mathfrak{L}_p$ and that $\mathfrak{R} = \mathfrak{R}_p$.

We can regard $N^1(X/W)$ as a subset of $N^1(Y/W)$ via ζ^*. Note that $(\Phi^p_\pi)^* D = (\Phi^p_{\pi'})^* D$ holds for all $D \in N^1(X/W)$. Because ζ is extremal, $N^1(X/W)$ has codimension 1 in $N^1(Y/W)$. By the projection formula [26, Proposition 2.5(c)], $N^1(X/W)$ must consist of the classes D such that

$$\deg_{C^p_k}(\Phi^p_\pi)^* D = 0.$$

The homomorphisms ϕ^p_π and $\phi^p_{\pi'}$ define isometric embeddings of $\mathfrak{L}_p \otimes \mathbb{R}$ into $\mathfrak{L}^p \otimes \mathbb{R}$ by Proposition 4.10. Hence, Proposition 6.3 implies that $\psi_{\mathbb{R}}^{-1}(N^1(Y/W))$ is the orthogonal complement $\mathbf{w}^\perp$ of $\mathbf{w}$ and that ϕ^p_π and $\phi^p_{\pi'}$ agree on $\psi_{\mathbb{R}}^{-1}(N^1(Y/W)) = \mathbf{w}^\perp$.

By Proposition 6.5, there exists a root $\mathbf{c} \in \mathfrak{R}$ with $\langle \mathbf{c}, \mathbf{c} \rangle = 2$ such that $\mathbf{c} \notin \mathbf{w}^\perp$. As $\mathbf{w} \in \mathfrak{L}^\dagger(\mathfrak{R})$ (by Proposition 4.10), we can assume that $\langle \mathbf{c}, \mathbf{w} \rangle = 1$. Hence, we have that $\mathbf{c} - \frac{\mathbf{w}}{\langle \mathbf{w}, \mathbf{w} \rangle} \in \mathbf{w}^\perp$. Define $\mathbf{r} = \phi^p_\pi(\mathbf{c})$ and $\mathbf{r}' = \phi^p_{\pi'}(\mathbf{c})$, which are elements of $\mathfrak{L}^p$ by Proposition 4.10. Note that for all $\mathbf{c}' \in \mathbf{w}^\perp$, we have that

$$\begin{aligned} \langle \mathbf{r} - \mathbf{r}', (\phi^p_\pi \otimes \mathbb{R})(\mathbf{c}') \rangle^p &= \langle \mathbf{r}, (\phi^p_\pi \otimes \mathbb{R})(\mathbf{c}') \rangle^p - \langle \mathbf{r}', (\phi^p_{\pi'} \otimes \mathbb{R})(\mathbf{c}') \rangle^p \\ &= \langle \mathbf{c}, \mathbf{c}' \rangle - \langle \mathbf{c}, \mathbf{c}' \rangle = 0 \end{aligned} \tag{3}$$

because ϕ^p_π and $\phi^p_{\pi'}$ are isometries that agree on $\mathbf{w}^\perp$.

As ϕ^p_π and $\phi^p_{\pi'}$ agree on $\mathbf{w}^\perp$, we also have that

$$\mathbf{r} - \mathbf{r}' = (\phi^p_\pi \otimes \mathbb{R}) \left(\frac{\mathbf{w}}{\langle \mathbf{w}, \mathbf{w} \rangle} \right) - (\phi^p_{\pi'} \otimes \mathbb{R}) \left(\frac{\mathbf{w}}{\langle \mathbf{w}, \mathbf{w} \rangle} \right).$$

As ϕ^p_π and $\phi^p_{\pi'}$ are isometries, it follows that

$$\begin{aligned} \langle \mathbf{r} - \mathbf{r}', \mathbf{r} - \mathbf{r}' \rangle^p &= \frac{\langle ((\phi^p_\pi - \phi^p_{\pi'}) \otimes \mathbb{R})(\mathbf{w}), ((\phi^p_\pi - \phi^p_{\pi'}) \otimes \mathbb{R})(\mathbf{w}) \rangle^p}{\langle \mathbf{w}, \mathbf{w} \rangle^2} \\ &= \frac{4\langle \mathbf{w}, \mathbf{w} \rangle - \langle ((\phi^p_\pi + \phi^p_{\pi'}) \otimes \mathbb{R})(\mathbf{w}), ((\phi^p_\pi + \phi^p_{\pi'}) \otimes \mathbb{R})(\mathbf{w}) \rangle^p}{\langle \mathbf{w}, \mathbf{w} \rangle^2} \end{aligned}$$

$$\le \frac{4}{\langle \mathbf{w},\mathbf{w}\rangle} - \frac{\langle ((\phi_\pi^p + \phi_{\pi'}^p)\otimes\mathbb{R})(\mathbf{w}), ((\phi_\pi^p + \phi_{\pi'}^p)\otimes\mathbb{R})(\mathbf{w})\rangle^p}{\langle \mathbf{w},\mathbf{w}\rangle^2} \tag{4}$$

$$\le \frac{4}{\langle \mathbf{w},\mathbf{w}\rangle} < \frac{9}{2},$$

where the second equality uses the identity

$$\langle a-b, a-b\rangle^p = 2\langle a,a\rangle^p + 2\langle b,b\rangle^p - \langle a+b, a+b\rangle^p.$$

Because $\mathfrak{L}^p$ is an even lattice, we must have that $\langle \mathbf{r}-\mathbf{r}', \mathbf{r}-\mathbf{r}'\rangle^p \in \{0,2,4\}$. We divide into cases based on the value of $\langle \mathbf{r}-\mathbf{r}', \mathbf{r}-\mathbf{r}'\rangle^p$ to show that ϕ_π^p and $\phi_{\pi'}^p$ are Weyl-related.

Case 1: $\langle \mathbf{r}-\mathbf{r}', \mathbf{r}-\mathbf{r}'\rangle^p = 0$. In this case, we have that $\mathbf{r} = \mathbf{r}'$. As $\mathbf{w}^\perp$ and $\mathbf{c}$ generate $\mathfrak{L}\otimes\mathbb{R}$, it follows that $\phi_\pi^p = \phi_{\pi'}^p$. In particular, ϕ_π^p and $\phi_{\pi'}^p$ are Weyl-related.

Case 2: $\langle \mathbf{r}-\mathbf{r}', \mathbf{r}-\mathbf{r}'\rangle^p = 2$. By Proposition 2.4, we have that $\mathbf{r}-\mathbf{r}' \in \mathfrak{R}^p$. We claim that $\phi_{\pi'}^p = s_{\mathbf{r}-\mathbf{r}'} \circ \phi_\pi^p$. As $\mathbf{w}^\perp$ and $\mathbf{c}$ generate $\mathfrak{L}\otimes\mathbb{R}$, it suffices to check that $s_{\mathbf{r}-\mathbf{r}'}$ restricts to the identity on $(\phi_\pi^p\otimes\mathbb{R})(\mathbf{w}^\perp)$ and that $s_{\mathbf{r}-\mathbf{r}'}(\mathbf{r}) = \mathbf{r}'$. The first part follows from (3). To see the second part, note that

$$\langle \mathbf{r},\mathbf{r}'\rangle^p = \frac{\langle \mathbf{r},\mathbf{r}\rangle^p + \langle \mathbf{r}',\mathbf{r}'\rangle^p - \langle \mathbf{r}-\mathbf{r}', \mathbf{r}-\mathbf{r}'\rangle^p}{2} = 1.$$

Hence, we have that $\langle \mathbf{r}, \mathbf{r}-\mathbf{r}'\rangle^p = 1$, so that

$$s_{\mathbf{r}-\mathbf{r}'}(\mathbf{r}) = \mathbf{r} - \langle \mathbf{r}, \mathbf{r}-\mathbf{r}'\rangle^p(\mathbf{r}-\mathbf{r}') = \mathbf{r} - (\mathbf{r}-\mathbf{r}') = \mathbf{r}'.$$

Therefore, ϕ_π^p and $\phi_{\pi'}^p$ are Weyl-related.

Case 3: $\langle \mathbf{r}-\mathbf{r}', \mathbf{r}-\mathbf{r}'\rangle^p = 4$. (4) implies that

$$\frac{4}{\langle \mathbf{w},\mathbf{w}\rangle} - \frac{\langle ((\phi_\pi^p + \phi_{\pi'}^p)\otimes\mathbb{R})(\mathbf{w}), ((\phi_\pi^p + \phi_{\pi'}^p)\otimes\mathbb{R})(\mathbf{w})\rangle^p}{\langle \mathbf{w},\mathbf{w}\rangle^2} \ge 4$$

$$\frac{\langle ((\phi_\pi^p + \phi_{\pi'}^p)\otimes\mathbb{R})(\mathbf{w}), ((\phi_\pi^p + \phi_{\pi'}^p)\otimes\mathbb{R})(\mathbf{w})\rangle^p}{\langle \mathbf{w},\mathbf{w}\rangle^2} \le \frac{4}{\langle \mathbf{w},\mathbf{w}\rangle} - 4 < \frac{1}{2}. \tag{5}$$

As $\mathbf{c} - \frac{\mathbf{w}}{\langle \mathbf{w}, \mathbf{w} \rangle} \in \mathbf{w}^\perp$, we have that

$$\begin{aligned} \langle \mathbf{r}_k^p, \mathbf{r} \rangle^p &= \frac{\langle \mathbf{r}_k^p, (\phi_\pi^p \otimes \mathbb{R})(\mathbf{w}) \rangle^p}{\langle \mathbf{w}, \mathbf{w} \rangle} + \left\langle \mathbf{r}_k^p, (\phi_\pi^p \otimes \mathbb{R}) \left(\mathbf{c} - \frac{\mathbf{w}}{\langle \mathbf{w}, \mathbf{w} \rangle} \right) \right\rangle^p \\ &= \frac{\langle \mathbf{r}_k^p, (\phi_{\pi'}^p \otimes \mathbb{R})(\mathbf{w}) \rangle^p}{\langle \mathbf{w}, \mathbf{w} \rangle}. \end{aligned}$$

As ϕ_π^p and $\phi_{\pi'}^p$ agree on $\mathbf{w}^\perp$, we also have that

$$\begin{aligned} \langle \mathbf{r}_k^p, \mathbf{r}' \rangle^p &= \frac{\langle \mathbf{r}_k^p, (\phi_{\pi'}^p \otimes \mathbb{R})(\mathbf{w}) \rangle^p}{\langle \mathbf{w}, \mathbf{w} \rangle} + \left\langle \mathbf{r}_k^p, (\phi_{\pi'}^p \otimes \mathbb{R}) \left(\mathbf{c} - \frac{\mathbf{w}}{\langle \mathbf{w}, \mathbf{w} \rangle} \right) \right\rangle^p \\ &= \frac{\langle \mathbf{r}_k^p, (\phi_{\pi'}^p \otimes \mathbb{R})(\mathbf{w}) \rangle^p}{\langle \mathbf{w}, \mathbf{w} \rangle} + \left\langle \mathbf{r}_k^p, (\phi_{\pi}^p \otimes \mathbb{R}) \left(\mathbf{c} - \frac{\mathbf{w}}{\langle \mathbf{w}, \mathbf{w} \rangle} \right) \right\rangle^p \\ &= \frac{\langle \mathbf{r}_k^p, (\phi_{\pi'}^p \otimes \mathbb{R})(\mathbf{w}) \rangle^p}{\langle \mathbf{w}, \mathbf{w} \rangle}. \end{aligned}$$

By the Cauchy–Schwarz inequality and (5), it follows that

$$\begin{aligned} \left| \langle \mathbf{r}_k^p, \mathbf{r} \rangle^p + \langle \mathbf{r}_k^p, \mathbf{r}' \rangle^p \right| &= \left| \frac{\langle \mathbf{r}_k^p, (\phi_\pi^p \otimes \mathbb{R})(\mathbf{w}) + (\phi_{\pi'}^p \otimes \mathbb{R})(\mathbf{w}) \rangle^p}{\langle \mathbf{w}, \mathbf{w} \rangle} \right| \\ &\leq \frac{\sqrt{2 \langle ((\phi_\pi^p + \phi_{\pi'}^p) \otimes \mathbb{R})(\mathbf{w}), ((\phi_\pi^p + \phi_{\pi'}^p) \otimes \mathbb{R})(\mathbf{w}) \rangle^p}}{\langle \mathbf{w}, \mathbf{w} \rangle} \\ &< 1. \end{aligned}$$

We have that $\langle \mathbf{r}_k^p, \mathbf{r} \rangle^p = 1$ by construction. Because $\langle \mathbf{r}_k^p, \mathbf{r}' \rangle^p \in \mathbb{Z}$, we must have that $\langle \mathbf{r}_k^p, \mathbf{r}' \rangle = -1$. It follows that $\langle \mathbf{r}_k^p, \mathbf{r} - \mathbf{r}' \rangle^p = 2$, so that

$$\langle \mathbf{r}_k^p - \mathbf{r} + \mathbf{r}', \mathbf{r}_k^p - \mathbf{r} + \mathbf{r}' \rangle^p = \langle \mathbf{r}_k^p, \mathbf{r}_k^p \rangle^p + \langle \mathbf{r} - \mathbf{r}', \mathbf{r} - \mathbf{r}' \rangle^p - 2 \langle \mathbf{r}_k^p, \mathbf{r} - \mathbf{r}' \rangle^p = 2.$$

Proposition 2.4 therefore guarantees that $\mathbf{r}_k^p - \mathbf{r} + \mathbf{r}' \in \mathfrak{R}^p$.

We claim that $\phi_{\pi'}^p = s_{\mathbf{r}_k^p - \mathbf{r} + \mathbf{r}'} \circ s_{\mathbf{r}_k^p} \circ \phi_\pi^p$. As $\mathbf{w}^\perp$ and $\mathbf{c}$ generate $\mathfrak{L} \otimes \mathbb{R}$, it suffices to check that $s_{\mathbf{r}_k^p - \mathbf{r} + \mathbf{r}'} \circ s_{\mathbf{r}_k^p}$ restricts to the identity on $(\phi_\pi^p \otimes \mathbb{R})(\mathbf{w}^\perp)$ and that $s_{\mathbf{r}_k^p - \mathbf{r} + \mathbf{r}'}(s_{\mathbf{r}_k^p}(\mathbf{r})) = \mathbf{r}'$. To see the first part, note that $\mathbf{r}_k^p$ and $\mathbf{r} - \mathbf{r}'$ are orthogonal to $(\phi_\pi^p \otimes \mathbb{R})(\mathbf{w}^\perp)$ by construction and (3), respectively. To see the second part, note that

$$\langle \mathbf{r}_k^p - \mathbf{r}, \mathbf{r}_k^p - \mathbf{r} + \mathbf{r}' \rangle^p = 2 - \langle \mathbf{r}', \mathbf{r}_k^p - \mathbf{r} + \mathbf{r}' \rangle^p = 1.$$

It follows that

$$\begin{aligned} s_{\mathbf{r}_k^p}(\mathbf{r}) &= \mathbf{r} - \langle \mathbf{r}, \mathbf{r}_k^p \rangle^p \mathbf{r}_k^p = \mathbf{r} - \mathbf{r}_k^p \\ s_{\mathbf{r}_k^p - \mathbf{r} + \mathbf{r}'}(s_{\mathbf{r}_k^p}(\mathbf{r})) &= \mathbf{r} - \mathbf{r}_k^p - \langle \mathbf{r} - \mathbf{r}_k^p, \mathbf{r}_k^p - \mathbf{r} + \mathbf{r}'^p \rangle (\mathbf{r}_k^p - \mathbf{r} + \mathbf{r}') \\ &= \mathbf{r} - \mathbf{r}_k^p + (\mathbf{r}_k^p - \mathbf{r} + \mathbf{r}') = \mathbf{r}'. \end{aligned}$$

Therefore, ϕ_π^p and $\phi_{\pi'}^p$ are Weyl-related.

We have proven that ϕ_π^p and $\phi_{\pi'}^p$ are Weyl-related in all three cases and that the cases exhaust all possibilities. The lemma follows. □

The next result shows—roughly—that if the sets of enhancement weights are independent of the choice of (very) good partial resolution, then the conclusion of Theorem 5.9(b) holds.

Proposition 7.4. *In Situation 5.4, if we have that*

$$\bigcup_{p \in W_{\text{sing}}(\mathbb{C})} \mathcal{W}_{\pi,\neq 0}^p = \bigcup_{p \in W_{\text{sing}}(\mathbb{C})} \mathcal{W}_{\pi',\neq 0}^p$$

as sets for all $\pi' \in \text{Good}(W)$, *then the conclusion of Theorem 5.9(b) holds.*

Proof. Define a set $\mathcal{W}$ by $\mathcal{W} = \bigcup_{p \in W_{\text{sing}}(\mathbb{C})} \mathcal{W}_{\pi',\neq 0}^p$ for any $\pi' \in \text{Good}(W)$. As $\mathcal{W}_{\pi',\neq 0}^p = \pm \mathcal{W}_{\pi',\neq 0}^{p,+}$ holds for all $p \in W_{\text{sing}}(\mathbb{C})$, we have that $\mathcal{W} = \pm \bigcup_{p \in W_{\text{sing}}(\mathbb{C})} \mathcal{W}_{\pi',\neq 0}^{p,+}$ as sets. In particular, for all $\pi' \in \text{Good}(W)$, the set of hyperplanes that that are orthogonal to elements of $\mathcal{W}_{\pi'}^{p,+}$ is the same as the set of hyperplanes that are orthogonal to elements of $\mathcal{W}$. By Theorem 5.5(b), it follows that, for all $\pi' \in \text{Good}(W)$, the cone $\psi_{\mathbb{R}}^{-1}(\overline{\text{Amp}}(Y'/W))$ is a closed chamber for the hyperplane arrangement in $\psi_{\mathbb{R}}^{-1}(\overline{\text{Mov}}(Y/X))$ consisting of hyperplanes orthogonal to the elements of $\mathcal{W}$. The conclusion of Theorem 5.9(b) follows by Theorem 5.5(c). □

We are now ready to complete the proof of Theorem 5.9.

Proof of Theorem 5.9. We first prove Part (a). Define a set $S \subseteq \text{Good}(W)$ by

$$S = \{\pi' \in \text{Good}(W) \mid \mathcal{W}_{\pi'}^p = \mathcal{W}_\pi^p \text{ for all } p \in W_{\text{sing}}(\mathbb{C})\}.$$

We show that S is closed under extremal flops. Let $\pi' : Y' \to W$ be a good partial resolution with $\mathcal{W}_{\pi'}^p = \mathcal{W}_\pi^p$ as multisets for all $p \in W_{\text{sing}}(\mathbb{C})$. Proposition 5.3 implies that π' is a very good partial resolution. Let $\zeta : Y' \to X$ be an extremal flopping contraction over W, let $\zeta' : Y'' \to X$ be the flop of ζ, and let π'' denote the composite $Y'' \to X \to W$. We prove that $\phi^p_{\pi'}$ and $\phi^p_{\pi''}$ are Weyl-related for all closed points $p \in W_{\text{sing}}$ by dividing into cases based on whether ζ contracts a curve above p.

Case 1: ζ does not contract any curve C^p_k. We have that $\phi^p_{\pi''} = \phi^p_{\pi'}$. In particular, $\phi^p_{\pi'}$ and $\phi^p_{\pi''}$ are Weyl-related.

Case 2: ζ contracts curve C^p_k. Let $\mathbf{w} = (\phi^p_\pi)^\dagger(\mathbf{r}^p_k) \in \mathcal{W}^p_{\pi',\neq 0} = \mathcal{W}^p_{\pi,\neq 0}$ denote the corresponding weight.

We claim that $\mathbf{w}$ is not proportional to any element of $\mathfrak{R}$. Suppose for sake of deriving a contradiction that $\mathbf{w}$ is proportional some $\mathbf{c} \in \mathfrak{R}$. Without loss of generality, we can assume that $\mathbf{w}$ is a positive multiple of $\mathbf{c}$. Let $D' \in \operatorname{Amp}(Y'/X)$ be arbitrary, so that ξ is a D'-flopping contraction and ξ' is the D'-flop of ξ. As $-D'$ is ξ-ample, Theorem 3.3 and Proposition 6.3 imply that $\langle -\psi_{\mathbb{R}}^{-1}(D'), \mathbf{c}\rangle < 0$, so that $\langle \psi_{\mathbb{R}}^{-1}(D'), \mathbf{c}\rangle > 0$. As $D' \in \overline{\operatorname{Mov}}(Y/X)$ by Theorem 3.37, Lemma 6.11 implies that $\mathbf{c}$ cannot be a positive linear combination of simple roots, so that $\mathbf{c}$ must be a negative root (for any polarization for which the simple roots are the $\mathbf{c}^{(j)}_i$).

Now let $D \in \operatorname{Amp}(Y/X)$ be arbitrary. By Theorem 3.3 and Proposition 6.3, we must have that $\langle \psi_{\mathbb{R}}^{-1}(D), \mathbf{c}\rangle < 0$. Because $\mathbf{c}$ is a negative root, Lemma 6.11 implies that $D \notin \overline{\operatorname{Mov}}(Y/X)$, contradicting the fact that ample divisors are movable (see, e.g., Theorem 3.37). Therefore, we can conclude that $\mathbf{w}$ cannot be proportional to any element of $\mathfrak{R}$.

Because π' is a very good partial resolution, it follows that Y' must be nonsingular above p. The hypothesis of the theorem implies that $\langle \mathbf{w}, \mathbf{w}\rangle > \frac{8}{9}$. Lemma 7.3 hence guarantees that $\phi^p_{\pi'}$ and $\phi^p_{\pi''}$ are Weyl-related.

The cases exhaust all possibilities, and hence we can conclude that $\phi^p_{\pi'}$ and $\phi^p_{\pi''}$ are Weyl-related for all closed points $p \in W_{\text{sing}}$. Lemma 7.2 implies that $\mathcal{W}^p_{\pi'} = \mathcal{W}^p_{\pi''}$ for all closed points $p \in W_{\text{sing}}$, and hence we have that $\pi'' \in S$. As π' and ζ were arbitrary, we have proven that S is closed under extremal flops over W. Theorem 3.40 hence implies that $S = \operatorname{Good}(W)$, proving Part (a).

In light of Proposition 7.4, Part (a) implies Part (b). □

8 A family of examples from the Weierstrass models of elliptic fibrations

In this section, we present applications of Theorems 5.5 and 5.9 to the KKMR decompositions for good partial resolutions of Weierstrass models. The examples that we consider feature non-Kodaira fibers in codimension 2, so that the results of Matsuki [57] do not apply. We consider examples based on Weierstrass models so that we can apply a generalized Tate's algorithm [4, 38, 77] to produce the desired Du Val singularity types in codimension 2.

In Section 8.1, we recall the general setup of Weierstrass models. In Section 8.2, we present the family of examples that we study, describing both the Weierstrass model and one example of a good partial resolution. In Section 8.3, we compute the enhancement homomorphisms for the good partial resolution described in Section 8.2 and apply our main results to characterize the KKMR decomposition.

8.1 Preliminaries on Weierstrass models

The general setup for Weierstrass models is the following situation.

Situation 8.1. Let B be a nonsingular variety, let $\mathcal{L} \in \operatorname{Pic} B$ be an invertible sheaf, and let $W \subseteq \mathbb{P}_B[\mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3} \oplus \mathcal{O}_B]$ be a closed subvariety of the form

$$V(y^2z + a_1xyz + a_3yz^2 - x^3 - a_2x^2z - a_4xz^2 - a_6),$$

where $[x; y; z]$ is the projective coordinate on $\mathbb{P}_B[\mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3} \oplus \mathcal{O}_B]$ and the coefficients a_i satisfy $a_i \in H^0(B, \mathcal{L}^{\otimes i})$ for $i = 1, 2, 3, 4, 6$. Suppose that the natural projection $\mu : W \to B$ is smooth at the generic point of B. We denote by $\sigma : B \to W$ the function $[0; 0; 1]$, and we write S for the image of σ.

Remark 8.2. We use the same convention for projective bundles as Eisenbud and Harris [15], so that points in $\mathbb{P}_B\mathcal{V}$ correspond to lines in the fibers of $\mathcal{V}$. Formally, if $\mathcal{V}$ is a locally free coherent sheaf on a scheme B, we let

$$\mathbb{P}_B\mathcal{V} = \operatorname{Proj}_B \operatorname{Sym}^\bullet \mathcal{V}^*,$$

where $\operatorname{Sym}^\bullet \mathcal{V}^*$ is the symmetric algebra of the dual of $\mathcal{V}$. This convention is dual to the one used by Grothendieck, who defined points in $\mathbb{P}_B\mathcal{V}$ to correspond to one-dimensional quotients of the fibers of $\mathcal{V}$.

The first result shows that W is normal and Gorenstein.

Lemma 8.3 ([69]). *In Situation 8.1, W is normal and Gorenstein.*

Proof. As W is a complete intersection, it is Gorenstein by [14, Corollary 21.19]. By construction, W is nonsingular in codimension 1. Serre's criterion [14, Theorem 18.15] implies that W is normal. □

We need to determine when W has cDV singularities. Whether this property holds depends in general on the orders to which the coefficients in the Weierstrass model vanish. To simplify the analysis, we reduce the defining equation. Formally, the defining equation of a Weierstrass model can be written in *reduced* form

$$y^2z - x^3 - fxz^2 - gz^3 = 0$$

by applying a linear change of coordinates on the projective bundle $\mathbb{P}_B[\mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3} \oplus \mathcal{O}_B]$. We say that the defining equation is *minimal* if, there exists a reduced form such that, for each closed point $p \in B$, either the coefficient f vanishes to order at most 3 at p or the coefficient g vanishes to order at most 5 at p.

Proposition 8.4. *In Situation 8.1, under minimality, W has cDV singularities.*

Proof. Let $p \in B$ be an arbitrary closed point, and let $H \subseteq B$ be a general hyperplane section through p. The proof of [69, Lemma 3.6] shows that $W \times_B \operatorname{Spec} \mathcal{O}_{H,p}$ has rational singularities (under minimality). As W is Gorenstein (by Lemma 8.3), so is $W \times_B \operatorname{Spec} \mathcal{O}_{H,p}$. Thus, $W \times_B \operatorname{Spec} \mathcal{O}_{H,p}$ has a Du Val singularity at every closed point. Hence, W has a cDV singularity at every closed point of $\mu^{-1}(p)$. As p was arbitrary, W must have cDV singularities. □

Remark 8.5. Proposition 8.4 was essentially proven by Nakayama [69]. However, Nakayama [69, Lemma 3.6] only asserted that W has rational Gorenstein singularities under minimality, when in fact W must have cDV singularities in this case.

We also need to determine when W is $\mathbb{Q}$-factorial, which we show occurs when the generic fiber has rank 0. Formally, let η denote the generic point of B. In Situation 4.1, note that W_η is an elliptic curve over $k(\eta)$ embedded in $\mathbb{P}^2_{k(\eta)}$ in Weierstrass form.

Proposition 8.6. *In Situation 8.1, if W_η has only finitely many $k(\eta)$-valued points, then W is $\mathbb{Q}$-factorial.*

Proof. Lemma 8.3 guarantees that W is normal. To prove the proposition, it remains to show that every divisor class is $\mathbb{Q}$-Cartier.

We first show that every divisor D whose support does not meet $\mu^{-1}(\eta)$ is Cartier. Let $D' = \mu_* D$, which must be a divisor on B by construction. As B is nonsingular, D' is Cartier. As μ is a contraction, we have that $D = \mu^* D'$, so that D is Cartier.

Let Z be an arbitrary prime divisor on W. We need to show that $n[Z]$ is Cartier for some $n \in \mathbb{Z}_{>0}$. If $\mu(Z) \neq B$, then the previous paragraph implies that $[Z]$ is Cartier. Hence, we can assume that $\mu(Z) = B$.

Using the group law on the elliptic curve W_η, we obtain a rational function $f \in k(W)$ such that the principal divisor of zeros and poles $(f)|_{W_\eta}$ is $[Z_\eta] - m[S_\eta] - [q]$, where $m \in \mathbb{Z}_{\geq 0}$ and q is a $k(\eta)$-valued point of W_η. As W_η has only finitely many $k(\eta)$-valued points, q must be a torsion point of the elliptic curve $W_\eta \to \operatorname{Spec} k(\eta)$. Hence, there exist $n \in \mathbb{Z}_{>0}$ and $g \in k(W)$ such that $(g)|_{W_\eta} = n[q] - n[S_\eta]$. As a result, we have that $(f^n g)|_{W_\eta} = n[Z_\eta] - (m+1)n[S_\eta]$. It follows that the support of $n[Z] - (m+1)n[S_\eta] - (f^n g)|_{W_\eta}$ does not meet $\mu^{-1}(\eta)$, so that the divisors $n[Z] - (m+1)n[S] - (f^n g)|_{W_\eta}$ must be Cartier. As S does not meet the singular locus of W, the divisor $[S]$ is also Cartier, and it follows that $n[Z]$ is Cartier as well.

Taking linear combinations, we see that W must be $\mathbb{Q}$-factorial. □

Remark 8.7. Proposition 8.6 is similar in spirit to results of Shioda [74] and Wazir [82], who constructed homomorphisms from the Mordell-Weil group to the Picard group. In essence, we exploit this homomorphism in our proof. Note that our argument works for bases B of arbitrary dimension.

In light of Proposition 8.6, W is $\mathbb{Q}$-factorial when the gauge group derived in the F-theory literature is semisimple. When the Mordell-Weil rank is positive, the F-theory literature considers a gauge algebra that is not semisimple—see Mayrhofer et al. [59], Lawrie et al. [52], and Morrison and Taylor [65]—to handle subtleties in the birational geometry. From our point of view, these subtleties are caused by the possibility that W is not $\mathbb{Q}$-factorial when the Mordell-Weil rank is positive.

8.2 A collision and its good partial resolution

Our family of examples is based on partial compactifications of a class of Weierstrass models introduced by Miranda [60]. In this class of Miranda models, the singular locus W_{sing} has two irreducible components Z_1 and Z_2, which are nonsingular curves. The Du Val singularity type at the generic points of Z_1 and Z_2 are B_n (with $n \geq 2$) and C_m (with $m \geq 4$) respectively. Hence, the gauge algebra is isomorphic to the $\mathfrak{sp}_n \oplus \mathfrak{so}_{2m+1}$, whose type is Langlands dual to $B_n \oplus C_m$. In the terminology of elliptic fibrations, we focus on the case in which the Kodaira singularity types below general points of Z_1 and Z_2 are I_{2n+1} and I^*_{m-3} respectively. As the gauge group is not simply-laced, these Kodaira fibers are *non-split* in the sense of Esole et al. [22, 23], in that one has to pass to field extensions of the function fields of Z_1 and Z_2 to obtain split Kodaira fibers—or, equivalently, Du Val singularities with simply-laced types—above the generic points of Z_1 and Z_2.

Tate's algorithm [77]—as generalized to elliptic threefolds by Bershadsky et al. [4] and Katz et al. [38]—provides *Tate forms* for the coefficients of the defining equation of the Weierstrass model that yield the Du Val singularity types described in the previous paragraph. These Tate forms give rise to instances of Situation 8.1. Formally, let B be a nonsingular surface. Let $S = V(s)$ and $T = V(t)$ be nonsingular curves on B that meet transversely, and let $\mathcal{L}$ be an invertible sheaf on B. If $m = 2k$ with $k \geq 2$, then the Tate form can be written as

$$y^2z - x^3 - b_2tx^2z - b_4s^{n+1}t^{k+1}xz^2 - b_6s^{2n+1}t^{2k}z^3, \tag{6}$$

where we have that

$$\begin{aligned}
&b_2 \in H^0(B, \mathcal{L}^{\otimes 2}(-[T])) \\
&b_4 \in H^0(B, \mathcal{L}^{\otimes 4}(-(n+1)[S] - (k+1)[T])) \\
&b_6 \in H^0(B, \mathcal{L}^{\otimes 6}(-(2n+1)[S] - 2k[T])).
\end{aligned}$$

In this case, the discriminant of the defining equation is

$$\begin{aligned}
\Delta = s^{2n+1}t^{2k+3}\left(4b_2^3b_6 - b_2^2b_4^2st + 27b_6^2s^{2n+1}t^{2k-3} - 18b_2b_4b_6s^{n+1}t^{k-1} + 4b_4^3s^{n+2}t^k\right) \\
\in H^0(B, \mathcal{L}^{\otimes 12}).
\end{aligned}$$

If $m = 2k+1$ with $k \geq 2$,[39] then the Tate form can be written as

$$y^2z - x^3 - b_2tx^2z - b_4s^{n+1}t^{k+1}xz^2 - b_6s^{2n+1}t^{2k+1}z^3, \tag{7}$$

where we have that

$$\begin{aligned} &b_2 \in H^0(B, \mathcal{L}^{\otimes 2}(-[T])) \\ &b_4 \in H^0(B, \mathcal{L}^{\otimes 4}(-(n+1)[S] - (k+1)[T])) \\ &b_6 \in H^0(B, \mathcal{L}^{\otimes 6}(-(2n+1)[S] - (2k+1)[T])). \end{aligned}$$

In this case, the discriminant of the defining equation is

$$\begin{aligned} \Delta = {} & s^{2n+1}t^{2k+4}\left(4b_2^3b_6 - b_2^2b_4^2s + 27b_6^2s^{2n+1}t^{2k-2} - 18b_2b_4b_6s^{n+1}t^{k-1} + 4b_4^3s^{n+2}t^{k-1}\right) \\ & \in H^0(B, \mathcal{L}^{\otimes 12}). \end{aligned}$$

Remark 8.8. We note that—due to the presence of non-split I_{2n+1} fibers—the Tate forms (6) and (7) are not the most general starting points for elliptic fibrations with gauge algebra $\mathfrak{sp}_n \oplus \mathfrak{so}_{2m+1}$. See Katz et al. [38, Section 4.10] for a detailed discussion of this point. For sake of simplicity, we consider only the Weierstrass models whose defining equations can be written in Tate forms similar to (6) and (7).

Some mild regularity conditions are needed for the Tate forms to give rise to the desired type of cDV singularities. In both cases, if

$$\begin{aligned} &\text{we have that } \ b_2b_6|_S \neq 0 \text{ and } \ b_2b_6|_T \neq 0 \text{ and that} \\ &\text{the divisor } V(\Delta s^{-2n-1}t^{-m-3}) \subseteq B \text{ is reduced and nonsingular,} \end{aligned} \tag{8}$$

then the singular locus of the Weierstrass model W is $V(x,y,st)$. Letting $Z_1 = V(x,y,s)$ and $Z_2 = V(x,y,t)$, if furthermore

$$\begin{aligned} &\text{the quadratic } z^2 - b_2t \text{ does not have a zero } z \in \mathbb{C}(S) \text{ and} \\ &\text{the quadratic } b_2x^2 + b_4s^{n+1}x + b_6s^{2n+1} \text{ does not have a zero } x \in \mathbb{C}(T), \end{aligned} \tag{9}$$

then the generalized Tate's algorithm [38] implies that W has a Du Val singularity of

[39] We ignore the case of $m = 3$ due to subtleties of Tate forms for type B_3 that arise from the triality symmetry of the Dynkin diagram of type D_4—see Esole et al. [22].

type C_n at the generic point of Z_1 and a Du Val singularity of type B_m at the generic point of Z_2. If (8) is satisfied and

$$V(b_2) \text{ and } S \text{ meet transversely and } V(b_2, t) = \emptyset, \tag{10}$$

then it can be straightforwardly verified that the defining equation is minimal, and hence Proposition 8.4 implies that W has cDV singularities. If furthermore

$$W_\eta \text{ has only finitely many } k(\eta)\text{-points, where } \eta \text{ is the generic point of } B, \tag{11}$$

then Proposition 8.6 guarantees that W is $\mathbb{Q}$-factorial. We summarize the discussion of this paragraph in the following lemma.

Lemma 8.9. *(a) If* (8) *and* (10) *are satisfied, then W has cDV singularities and $W_{\text{sing}} = Z_1 \cup Z_2$, where $Z_1 = V(x, y, s)$ and $Z_2 = V(x, y, t)$.*

(b) Letting η_1 and η_2 denote the generic points of Z_1 and Z_2, respectively, if furthermore (9) *is satisfied, then the local rings $\mathcal{O}_{W,\eta_1}$ and $\mathcal{O}_{W,\eta_2}$ have Du Val singularities of types B_n and C_m, respectively.*

(c) If (11) *is satisfied, then W is $\mathbb{Q}$-factorial.*

We now describe a good partial resolution of W–under further regularity conditions. As resolutions involve blowing up repeatedly, we need to fix notation for the defining equations of Cartier divisors that lie above the centers of blow-ups. Our notation follows Esole and Yau [17, Section 4.1] and Lawrie and Schäfer-Nameki [51, Section 2.3]. Given an integral scheme X and nonzero functions $s_i \in H^0(X, \mathcal{L}_i)$ for $1 \leq i \leq \ell$, we write $\mathrm{Bl}_{s_1,\ldots,s_\ell|e}$ for the blow-up of X at $V(s_1, \ldots, s_\ell)$. We let e be such that $V(e) = f^{-1}(V(s_1, \ldots, s_\ell))$, where $f : \mathrm{Bl}_{V(s_1,\ldots,s_n)} X \to X$ is the projection. By abuse of notation, we write s_i for $e^{-m_i} f^*(s_i)$—where m_i is the order to which s_i vanishes at $V(s_1, \ldots, s_\ell)$—so that $V(s_i)$ is the strict transform of $V(s_i)$ under f^{-1}.

Lemma 8.10. *Under* (8), (9), (10), *and* (11), *suppose that*

$$V(b_6, s, t) = \emptyset. \tag{12}$$

Consider the sequence of blow-ups

$$\mathrm{Bl}_{x,y,s_{n-1}|s_n} \cdots \mathrm{Bl}_{x,y,s_2|s_3} \mathrm{Bl}_{x,y,s_1|s_2} \mathrm{Bl}_{x,y,s|s_1}$$
$$\mathrm{Bl}_{u_k,v_{k-1}|w} \mathrm{Bl}_{x,v_{k-1}|u_k} \cdots \mathrm{Bl}_{x,v_2|u_3} \mathrm{Bl}_{y,u_2|v_2} \mathrm{Bl}_{x,v_1|u_2} \mathrm{Bl}_{y,u_1|v_1} \mathrm{Bl}_{x,y,t|u_1} W.$$

if $m = 2k$ and the sequence of blow-ups

$$\mathrm{Bl}_{x,y,s_{n-1}|s_n} \cdots \mathrm{Bl}_{x,y,s_2|s_3} \mathrm{Bl}_{x,y,s_1|s_2} \mathrm{Bl}_{x,y,s|s_1}$$
$$\mathrm{Bl}_{u_k,v_k|w} \mathrm{Bl}_{y,u_k|v_k} \cdots \mathrm{Bl}_{x,v_2|u_3} \mathrm{Bl}_{y,u_2|v_2} \mathrm{Bl}_{x,v_1|u_2} \mathrm{Bl}_{y,u_1|v_1} \mathrm{Bl}_{x,y,t|u_1} W$$

if $m = 2k+1$. In either case, the sequence of blow-ups yields a good partial resolution of W that is nonsingular away from the inverse image of $V(b_6, x, y, s)$ and has Cartier Cartan divisor classes $[Z_i^{(j)}]$.

To prove Lemma 8.10, we first provide explicit sections of invertible sheaves that cut out the Cartan divisors. By considering the singularities of the Cartan divisors, we next show first show that the composite of blow-ups yields a variety with only finitely singular points and that the singular points all lie over $V(b_6, x, y, s)$. The blow-up is Gorenstein by construction, and Serre's criterion therefore implies that the blow-up is therefore normal. Hence, the sequence of blow-ups defines a partial resolution. We then apply Proposition 3.6 to show that the partial resolution is crepant. In light of Lemma 8.9, W is $\mathbb{Q}$-factorial and has cDV singularities, and Proposition 5.1 therefore guarantees that the partial resolution is good.

Proof sketch. We first explicitly show that the Cartan divisor classes are all Cartier. By reordering the curves if necessary, we can assume that $-M(\mathcal{O}_{W,\eta_i})$ is one of the Killing matrices of Table 1 on page 17 for $i = 1, 2$. It can be straightforwardly verified that $Z_1^{(j)} = V(s_j)$ for $1 \leq j \leq n$. For $m = 2k$, it can be verified that

$$Z_2^{(1)} = V\left(\frac{\prod_{i=1}^{k-1} v_i}{\prod_{i=1}^{k-1} u_i}\right)$$
$$Z_2^{(2)} = V(u_1)$$
$$Z_2^{(2j-1)} = V\left(\frac{\prod_{i=j}^{k-1} u_j}{\prod_{i=j}^{k-1} v_j}\right) \qquad \text{for } 2 \leq j \leq k-1$$

$$
\begin{aligned}
Z_2^{(2j)} &= V\left(\frac{\prod_{i=j}^{k-1} v_j}{\prod_{i=j+1}^{k-1} u_j}\right) && \text{for } 2 \le j \le k-1 \\
Z_2^{(2k-1)} &= V(w) \\
Z_2^{(2k)} &= V(u_m).
\end{aligned}
$$

For $m = 2k+1$, it can also be verified that

$$
\begin{aligned}
Z_2^{(1)} &= V\left(\frac{\prod_{i=1}^{k-1} v_i}{\prod_{i=1}^{k} u_i}\right) \\
Z_2^{(2)} &= V(u_1) \\
Z_2^{(2j-1)} &= V\left(\frac{\prod_{i=j}^{k} u_j}{\prod_{i=j}^{k-1} v_j}\right) && \text{for } 2 \le j \le k \\
Z_2^{(2j)} &= V\left(\frac{\prod_{i=j}^{k-1} v_j}{\prod_{i=j+1}^{k} u_j}\right) && \text{for } 2 \le j \le k-1 \\
Z_2^{(2k)} &= V(w) \\
Z_2^{(2k+1)} &= V(v_m).
\end{aligned}
$$

In either case, all of the Cartan divisor classes are Cartier, as claimed.

We denote the composite of the sequence of blow-ups by $\pi : Y \to W$. Define a variety Y' over $W' = \mathbb{P}_B[\mathfrak{L}^{\otimes 2} \oplus \mathfrak{L}^{\otimes 3} \oplus \mathcal{O}_B]$ by

$$
\begin{aligned}
Y' = \mathrm{Bl}_{x,y,s_{n-1}|s_n} \cdots \mathrm{Bl}_{x,y,s_2|s_3} \mathrm{Bl}_{x,y,s_1|s_2} \mathrm{Bl}_{x,y,s|s_1} \\
\mathrm{Bl}_{u_k,v_{k-1}|w} \mathrm{Bl}_{x,v_{k-1}|u_k} \cdots \mathrm{Bl}_{x,v_2|u_3} \mathrm{Bl}_{y,u_2|v_2} \mathrm{Bl}_{x,v_1|u_2} \mathrm{Bl}_{y,u_1|v_1} \mathrm{Bl}_{x,y,t|u_1} W'
\end{aligned}
$$

if $m = 2k$ and

$$
\begin{aligned}
Y' = \mathrm{Bl}_{x,y,s_{n-1}|s_n} \cdots \mathrm{Bl}_{x,y,s_2|s_3} \mathrm{Bl}_{x,y,s_1|s_2} \mathrm{Bl}_{x,y,s|s_1} \\
\mathrm{Bl}_{u_k,v_k|w} \mathrm{Bl}_{y,u_k|v_k} \cdots \mathrm{Bl}_{x,v_2|u_3} \mathrm{Bl}_{y,u_2|v_2} \mathrm{Bl}_{x,v_1|u_2} \mathrm{Bl}_{y,u_1|v_1} \mathrm{Bl}_{x,y,t|u_1} W'
\end{aligned}
$$

if $m = 2k+1$. Because s and t define nonsingular curves in B, the centers of the blow-ups are nonsingular. Hence Y' is obtained from the nonsingular variety W' by blow-ups at nonsingular centers, so that Y' must be nonsingular. The variety Y is the proper transform of W in Y', and is cut out in Y' by section f defined by

$$\left(\prod_{i=1}^{k-1} v_i\right) y^2 z - \left(\prod_{i=1}^{k-1} u_i\right) \left(\begin{array}{l} \left(\prod_{i=1}^{n} s_i\right)\left(\prod_{i=2}^{k-1}(u_i v_i)\right) u_k^2 w^2 x^3 + b_2 t u_k x^2 z \\ +b_4 s^{n+1}\left(\prod_{i=1}^{n} s_i^{n+1-i}\right) t^{k+1}\left(\prod_{i=1}^{k-1}(u_i v_i)^{k-i}\right) u_k x z^2 \\ +b_6 s^{2n+1}\left(\prod_{i=1}^{n} s_i^{2n+1-2i}\right) t^{2k}\left(\prod_{i=1}^{k-1}(u_i v_i)^{2k-1-2i}\right) z^3 \end{array}\right)$$

if $m = 2k$ and

$$\left(\prod_{i=1}^{k} v_i\right) y^2 z - \left(\prod_{i=1}^{k} u_i\right) \left(\begin{array}{l} \left(\prod_{i=1}^{n} s_i\right)\left(\prod_{i=2}^{k}(u_i v_i)\right) w^2 x^3 + b_2 t x^2 z \\ +b_4 s^{n+1}\left(\prod_{i=1}^{n} s_i^{n+1-i}\right) t^{k+1}\left(\prod_{i=1}^{k-1}(u_i v_i)^{k-i}\right) x z^2 \\ +b_6 s^{2n+1}\left(\prod_{i=1}^{n} s_i^{2n+1-2i}\right) t^{2k+1}\left(\prod_{i=1}^{k-1}(u_i v_i)^{2k-2i}\right) z^3 \end{array}\right)$$

if $m = 2k + 1$.

For ease of notation, define an integer

$$\ell = \begin{cases} k-1 & \text{if } m = 2k \\ k & \text{if } m = 2k+1 \end{cases}.$$

On Y', we have that

$$\pi^* s = s \prod_{i=1}^{n} s_i$$
$$\pi^* t = t w^2 \left(\prod_{i=1}^{k} u_i\right)\left(\prod_{i=1}^{\ell} v_i\right).$$

Define a section

$$r = \begin{cases} u_k & \text{if } m = 2k \\ v_k & \text{if } m = 2k+1. \end{cases}$$

We next show that $Y_{\text{sing}} \subseteq V(b_6, x, \pi^* s)$ as sets. As the centers of the blow-ups all lie over $V(x, y, st)$, the morphism π must be an isomorphism over the nonsingular locus of W by Lemma 8.9. Hence, $\pi^{-1}(W \setminus W_{\text{sing}})$ must be nonsingular.

We check nonsingularity by considering the singularities of Cartan divisors. The Cartan divisor $Z_1^{(j)}$ is cut out in the nonsingular variety $V(s_j) \subseteq Y$ by the restriction of f, which can be written as

$$f_1^{(j)} = \left(\prod_{i=1}^{\ell} v_i\right) y^2 z - b_2 t \left(\prod_{i=1}^{k} u_i\right) x^2.$$

Note that $V((\pi^* t)/r, s_j) = \emptyset$ on Y' for $1 \leq j \leq n$ by construction. Hence, the section $f_1^{(j)}$ vanishes to order at most 1 except on $V(r, xy, s_j) \cup V(x, y, s_j)$, so that $Z_1^{(j)}$ is nonsingular away from $V(r, xy, s_j) \cup V(x, y, s_j)$. As $Z_1^{(j)}$ is a Cartier divisor for all $1 \leq j \leq n$, it follows that Y is nonsingular on $V(s_j) \setminus (V(r, xy, s_j) \cup V(x, y, s_j))$ for all $1 \leq j \leq n$. Because S and T meet transversely on B, the section f vanishes to order only 1 at every closed point in $V(r, xy, s_j) \subseteq Y'$. For $1 \leq j \leq n-1$, we have that $V(x, y, s_j) = \emptyset$ on Y' by construction. Note also that the section f vanishes to order 1 at every closed point in $V(x, y, s_n) \setminus V(x, y, s_n, b_6)$. It follows that Y is nonsingular along

$$V\left(\prod_{i=1}^{n} s_i\right) \setminus V(x, y, s_n, b_6).$$

It can also be verified straightforwardly that Y is nonsingular along $V(s) \setminus V(\pi^* t)$.

A similar argument shows that Y is nonsingular along

$$(V(\pi^* t) \setminus V(b_2, x, \pi^* t)) = V(f^* t),$$

where the second equality is due to (10). Therefore, we have that $Y_{\text{sing}} \subseteq V(b_6, x, \pi^* s)$ as sets, so that Y_{sing} is a finite subset of $\pi^{-1}(V(b_6, x, y, s))$. In particular, Y is nonsingular in codimension 2.

Because Y is the prime divisor $V(f)$ in the nonsingular variety Y', the variety Y must be Gorenstein by [14, Corollary 21.19]. As Y is nonsingular in codimension 2, it is nonsingular is codimension 1 and hence normal by Serre's criterion [14, Theorem 18.15]. Therefore, π is a partial resolution.

As W is normal (by Lemma 8.3), applying Proposition 3.6 repeatedly shows that f is crepant. Note that f is projective by construction. Because W is $\mathbb{Q}$-factorial and has cDV singularities (by Lemma 8.9) and Y is nonsingular in codimension 2 (as we have shown in the course of this proof), Proposition 5.1 implies that π is good. □

8.3 Using the main results to find the KKMR decomposition

As shown by Miranda [60, Section 11], there is a crepant resolution in a neighborhood of $V(s,t)$ whose fiber above $V(s,t)$ of the partial resolution is an incomplete I^* fiber—a non-Kodaira fiber. More precisely, in the Miranda [60] resolution, the dual graph fiber above any point in the intersection $Z_1 \cap Z_2$ is an incomplete type D Dynkin diagram. In particular, there are non-Kodaira fibers, so that the results of Matsuki [57, Section II-2] do not apply.[40] Nevertheless, we are able to completely describe the KKMR decomposition using Theorems 5.5 and 5.9.

Corollary 8.11. *Under* (8)*,* (9)*,* (10)*,* (11)*, and* (12)*, the following conclusions hold.*

(a) The good partial resolution given in Lemma 8.10 is very good.

(b) The KKMR decomposition (1) *of Theorem 5.5 is given by the decomposition* $\mathrm{I}(\mathfrak{sp}_n \oplus \mathfrak{so}_{2n+1}, (\mathbf{vec}, \mathbf{vec}))$*, where* $(\mathbf{vec}, \mathbf{vec})$ *is the bifundamental representation, i.e., the tensor product of the defining (first fundamental) representation of* $\mathfrak{sp}_n$ *and the defining (vector) representation of* $\mathfrak{so}_{2n+1}$*.*

To prove Corollary 8.11, we determine the sets of enhancement weights for the good partial resolution given in Lemma 8.10. We show that all enhancement weights below singular points are proportional to roots, and that the weights that are not proportional to roots that appear anywhere are precisely the weights of the bifundamental representation. The corollary then follows from Corollary 5.10.

Proof sketch. By Lemma 8.10, the sequence of blow-ups defined in Lemma 8.10 gives a good partial resolution, which we denote by $\pi : Y \to W$. Note also that Y is nonsingular above $Z_1 \cap Z_2$ because $V(b_6, s, t) = \emptyset$ on B.

We compute the enhancement weights of π at every singular point. For all closed points $p \in Z_1 \setminus Z_2$ with $p \notin V(b_2 b_6)$, it can be verified straightforwardly that the multiset $\mathcal{W}^p_{\pi,\neq 0}$ consists of 2 (resp. 1) copies of every short (resp. long) element of $\mathfrak{R}_1$, as well as two copies of every weight of the defining (first fundamental) representation $\mathbf{vec}$ of $\mathfrak{sp}_n$. Here, we say that a root is *short* if it has length $\sqrt{2}$ and *long* otherwise. For $p \in V(b_6, x, y, s)$, it can be verified that the multiset $\mathcal{W}^p_{\neq 0}$ consists of 2 (resp. 1) copies of every short (resp. long) element of $\mathfrak{R}_1$, as well as 4 copies of every weight

[40] It is a consequence of [57, Theorem II-2-1-1] that all crepant resolutions have Kodaira fibers under the hypotheses of [57, Theorem II-2-1-1].

of the defining (first fundamental) representation **vec** of $\mathfrak{sp}_n$. For $p \in V(b_2, x, y, s)$, it can be verified that the multiset $\mathcal{W}^p_{\pi,\neq 0}$ consists of 4 (resp. 1) copies of every short (resp. long) element of $\mathfrak{R}_1$, as well as 6 copies of every weight of the defining (first fundamental) representation **vec** of $\mathfrak{sp}_n$. Similar characterizations of $\mathcal{W}^p_\pi$ hold for all closed points $p \in Z_2 \setminus Z_1$ and all closed points $p \in V(b_2, x, y, s)$. In all of these cases—which together comprise all closed points $p \in (Z_1 \cup Z_2) \setminus (Z_1 \cap Z_2)$—all of the enhancement weights are proportional to roots. Because π is nonsingular above $Z_1 \cap Z_2$, we have therefore proven Part (a).

We now compute $\mathcal{W}^p_\pi$ for $p \in Z_1 \cap Z_2$. Let $p \in Z_1 \cap Z_2$ be arbitrary, and let H^p denote the inverse image in W of a general hyperplane section through the image of p in B. By construction, we can take H^p to meet Z_1 and Z_2 transversely at p. Tate's algorithm [77] guarantees that $\mathcal{O}_{H^p,p}$ has a Du Val singularity of type D_{m+2n+2}. It can be verified straightforwardly from Proposition 6.2 that ϕ^p_π is given by

$$\begin{aligned}
\phi^p_\pi(\mathbf{c}_1^{(j)}) &= \mathbf{r}^p_{m+2j-1} + 2\mathbf{r}^p_{m+2j} + \mathbf{r}^p_{m+2j+1} && \text{for all } 1 \le j \le n-1 \\
\phi^p_\pi(\mathbf{c}_1^{(n)}) &= \mathbf{r}^p_{m+2n-1} + 2\mathbf{r}^p_{m+2m} + \mathbf{r}^p_{m+2n+1} + \mathbf{r}^p_{m+2n+2} && \\
\phi^p_\pi(\mathbf{c}_2^{(j)}) &= \mathbf{r}^p_j && \text{for all } 1 \le j \le m-1 \\
\phi^p_\pi(\mathbf{c}_2^{(m)}) &= 2\sum_{k=m}^{m+2n} \mathbf{r}^p_k + \mathbf{r}^p_{m+2n+1} + \mathbf{r}^p_{m+2n+2}. &&
\end{aligned}$$

Here, we are ordering the simple roots so that the Killing matrix is exactly one of the matrices given in Table 1 on page 17—without having to permute (simultaneously) rows and columns. A simple calculation from the formula for ϕ^p_π shows that $\mathcal{W}^p_{\pi,\neq 0} \setminus \mathbb{R}\mathfrak{R}$ consists of 2 copies of each weight of the bifundamental representation.

Taking unions over all p, we see that the enhancement weights that appear that are not proportional to roots are precisely the weights of the bifundamental representation. Part (b) therefore follows from Corollary 5.10. □

Corollary 8.11 illustrates how our main results can be used to characterize the KKMR decomposition of a relative minimal model of a resolution of a $\mathbb{Q}$-factorial threefold with cDV singularities without having to compute all of the relative minimal models. Indeed, in the course of the proof of Corollary 8.11, we only had to deal with one good partial resolution—the one constructed in Lemma 8.10. Yet, we were able to leverage Theorems 5.5 and 5.9—in the form of Corollary 5.10—to characterize the KKMR decomposition.

A Proof of Proposition 3.16

We first prove Part (a). By [54, Theorem 4.1], there exists a sequence of schemes

$$X = X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 = \operatorname{Spec} R$$

such that X_{i+1} is obtained from X_i by blowing up a singular, closed point $p_i \in X_i$ for all i. Because R is of essentially finite type over a field, R is a Nagata ring by [76, Tags 0335 and 032U] and is excellent by [76, Tags 07QU and 07QW]. The completion of R is normal by [76, Tag 0C23]. As a result, [76, Tag 0BGC] applies. [76, Tag 0BGB] implies that X_i/X_{i-1} has a dualizing sheaf $\omega_{X_i/X_{i-1}} \simeq \mathcal{O}_{X_i}$ for all i. By Proposition 3.11, it follows that $X/\operatorname{Spec} R$ has a dualizing sheaf $\omega_{X/\operatorname{Spec} R} \simeq \mathcal{O}_X$, as desired.

We next prove Part (b), continuing with the preceding notation. By the definition of minimality (Theorem 3.15), f must factor through g via some morphism $h : Y \to X$. Note that h must be proper and birational because f and g are both proper and birational. As Y and X are regular two-dimensional Noetherian schemes, h must factor as a sequence of blow-ups at closed points by [76, Tag 0C5R].

Write h as a composite $h_2 \circ h_1$, where $h_1 : Y \to T$ and $h_2 : T \to X$ are proper birational morphisms, T is regular, and h_1 is given by a blow-up at a regular, closed point $p \in T$. Note that h_1 is a morphism between regular schemes, and is hence a local complete intersection morphism. As a result, T/X has a dualizing sheaf that is invertible by [76, Tags 0BR0 and 0BRT]. By Part (a) and Proposition 3.11, $T/\operatorname{Spec} R$ also has a dualizing sheaf that is invertible.

Applying the results of [76, Tag 0AU3] on the dualizing sheaves of blow-ups regular two-dimensional Noetherian schemes, we see that $\omega_{\mathbb{P}^1_T} \simeq \pi^*\omega_T \otimes \mathcal{O}_{\mathbb{P}^1_T}(-2)$, where $\pi : \mathbb{P}^1_T \to T$ is the projection. Regarding Y as an effective Cartier divisor on $\mathbb{P}^1_T$ and applying the adjunction formula for dualizing sheaves [76, Tag 0AU3], we have that $\omega_{Y/T} \simeq j^*\mathcal{O}_{\mathbb{P}^1_T}(-1)$. By Proposition 3.11, it follows that $\omega_{Y/\operatorname{Spec} R} \simeq j^*\mathcal{O}_{\mathbb{P}^1_T}(-1) \otimes h_1^*\omega_{T/\operatorname{Spec} R}$. Taking E to be the exceptional curve of h_1, we have that $\omega_{Y/\operatorname{Spec} R}\big|_E \simeq j^*_{\mathbb{P}^1_T}\mathcal{O}(-1)\big|_E$, which defines a non-torsion class in $\operatorname{Pic} E$. As $j^* : \operatorname{Pic} X \to \operatorname{Pic} E$ is well-defined, the class of $\omega_{Y/\operatorname{Spec} R}$ in $\operatorname{Pic} Y$ must also be non-torsion, as desired.

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