Complex Singularities and Twist Quantum Fields

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O blinding hour, O holy, terrible day,
When first the shaft into his vision shone
Of light anatomized! Euclid alone
Has looked on Beauty bare. Fortunate they
Who, though once only and then but far away,
Have heard her massive sandal set on stone.

—MILLAY.

Abstract

In this thesis, we give an expository overview of the work of John Milnor concerning the topology of functions $f\colon \mathbf{C}^m \to \mathbf{C}$ of several complex variables with a singular point at the origin. In particular, we focus on a certain fibration of the set $S_\epsilon^{2n+1} \setminus f^{-1}(0)$ over the circle, where S_ϵ^{2n+1} is a hypersphere of sufficiently small radius about the origin. We relate this work to more recent discoveries in quantum field theory by Arthur Jaffe and, his graduate student, Robert Martinez, that elucidates and classifies the moduli space of twist-regularized, generalized Wess–Zumino models of supersymmetric quantum field theory on the spacetime torus.

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Da steh' ich nun, ich armer Thor!
Und bin so klug als wie zuvor.

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1 Introduction

This thesis is comprised of two parts, the mathematical and the physical. The principal question with which we concern ourselves in the mathematical section is the topological analysis of the behavior of complex-valued functions of several complex variables $f\colon \mathbf{C}^n \to \mathbf{C}$ that satisfy both $f(0) = 0$ and $f'(0) = 0$; we say that f has a singular point at the origin. Such questions were originally considered in a cumbersome fashion by Brauner [1] in the course of an investigation of the the branching behavior of multi-valued functions of z_0 in some other variables $z_1, \ldots, z_n$ defined implicitly in the form $f(z_0, z_1, \ldots, z_n) = 0$. The branching behavior of such functions corresponds to that of the corresponding elliptic discriminants, which can have singularities at the origin.

In the case $n = 2$, Brauner analyzed the behavior of singularities of a discriminant function $f\colon \mathbf{C}^2 \to \mathbf{C}$ by taking the intersection of $f^{-1}(0)$ with a small sphere $S^3_\epsilon(0)$ about the origin with radius ϵ and $2n$ real dimensions. This approach was later refined by Kähler [9], who adapted Brauner's approach to use polydisks, or the boundaries of sets of the form $\{|z_1| < \eta_1, \ldots, |z_n| < \eta_n\} \subset \mathbf{C}^n$ for some positive quantities η. Kähler thereby sidestepped much of the algebraic complication that Brauner's use of the stereographic projection introduced. One of the results of Brauner and Kähler, since generalized to larger classes of polynomials, is that under this procedure the function $f(x, y) = x^2 + y^3$ gives rise to the trefoil knot (Figure 1).

Milnor made a fuller application of the tools of algebraic topology to the study of singularities of complex functions. He showed that the $2n$-real-dimensional complement of the knot in S^{2n+1}_ϵ could be given the structure of a fibration over the circle S^1 by associating to every point z the complex argument of $f(z)$. Furthermore, a topological invariant of any one of the resulting fibers—the n-th Betti number, or the rank of the middle homology group H_n—coincides with several algebraic invariants.

In the second part, we discuss the application of complex algebraic geometry, commutative algebra and algebraic topology to problems in physics, most notably quantum field theory on the spacetime torus. In a series of papers, Arthur Jaffe computed the elliptic genus of the twist-regularized Wess–Zumino model with a superpotential given by a weighted homogeneous polynomial. He proves the existence of the elliptic genus and shows that it satisfies several surprising transformation properties. In his epic doctoral dissertation, Robert Martinez, Jaffe's most recent graduate student, extended some of his advisor's results and proved incredibly deep connections between these QFT models and Milnor's study of singularities of complex hypersurfaces, creating a bridge between hitherto disparate realms of mathematics and physics.

2 Beginnings: Brauner and Kähler

The topological study of the zeroes of complex functions near a singular point was initiated by Brauner, who studied functions of two complex variables $f(w,z) = w^2+z^3$. Intuitively, as the map $w \to w^2$ (resp. $z \mapsto z^3$) wraps a small disk in one complex plane around itself twice (resp. thrice), a function that combines the wrappings $w \mapsto w^2$ and $z \mapsto z^3$ should be some kind of entanglement of the two, though the distance from this intuitive notion to a full mathematical understanding is considerable.[1]

Brauner [1] makes this notion precise in the following way. Consider complex numbers κ, ϵ, where $|\kappa| \ll |\epsilon| \ll 1$, and take the intersection of the complex manifold $f^{-1}(\kappa)$ with the 3-sphere $S^3_\epsilon = \{(w,z) \in \mathbf{C}^2 | w\bar{w} + z\bar{z} = \epsilon^2\}$. These are respectively manifolds of two and three real dimensions in an ambient space $\mathbf{C}^2 \cong \mathbf{R}^4$ of four real dimensions; their intersection should therefore ordinarily have dimension 1. In fact, as Brauner illustrates, this intersection is precisely a *trefoil knot*.

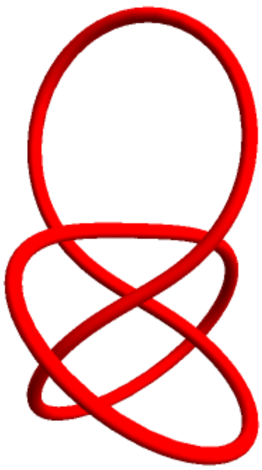

Figure 1: A trefoil knot (from [13]).

Brauner is principally interested in the branching behavior of implicitly defined, multi-valued functions, such as when z_0 is a function of complex variables $z_1, \ldots, z_n$ defined by a relationship $f(z_0, \ldots, z_n) = 0$; he finds a relationship between this and the topology of singular points of the corresponding elliptic discriminant. He begins his exposition by noting that according to Weierstrass' Preparation Theorem, any analytic function of complex variables $z_0, \ldots, z_n$ with a branch point at the origin can be written as the product of a function of the form

$$z_0^k + g_{k-1}(z_1, \ldots, z_n) z_0^{k-1} + \ldots + g_0(z_1, \ldots, z_n) \tag{1}$$

[1]Brieskorn and Knörrer [3], p. 393 point out that the solution set to $x^2 = y^3$, intersected with the polydisk considered in [9], is a type of *braid*.

where the coefficients g_i are analytic, with some other function that is not zero at the singular point in question.

The type of branch point depends only on the first term in this product, the polynomial in z_0. We can therefore assume without loss of generality that any arbitrary analytic function that we wish to study has the form of a monic polynomial.

Now let $D(z_0, \ldots, z_n)$ be the discriminant of this function. At most points, D may be divided as follows:

$$D(z_0, \ldots, z_n) = \left(\sum_i a_i z_i + \sum_{ij} a_{ij} z_i z_j + \cdots \right)^d L(z_0, \ldots, z_k), \tag{2}$$

where at most one of the coefficients a_i is nonzero—assume without loss of generality that this is a_0—and L is nonzero in a neighborhood of the singularity. This means that D has a nonzero first directional derivative in at least one direction. Brauner classifies all points in $\mathbf{C}^k$ according to whether the discriminant at that point may be written in such a form. The points at which this is possible are "Type I" branch points and include all regular points of the discriminant; the other points are called "Type II," and correspond to locations in which the the discriminant has a singularity.

About a Type I point, a bijective[2] change of coordinates $\xi = \sum_i a_i z_i + \sum_{ij} a_{ij} z_i z_j + \cdots$ and $z'_i = z_i$ for $1 \leq i \leq n$ allows rewriting the polynomial as

$$z_0^k + f_{k-1}(\xi, z'_1, \ldots, z'_n) z_1^{k-1} + \cdots + f_0(\xi, z'_1, \ldots, z'_n). \tag{3}$$

This has a discriminant of the form $\xi^d k(\xi, z'_1, \ldots, z'_n)$, where k is nonzero in a neighborhood of the origin $\xi = z'_1 = \ldots = z'_n = 0$. If we hold the variables z'_i fixed, then the equation

$$z_0^k + f_{k-1}(\xi, z'_1, \ldots, z'_k) z_1^{n-1} + \cdots + f_0(\xi, z'_1, \ldots, z'_n) = 0 \tag{4}$$

gives z_0 implicitly as an algebraic function of ξ and the z_i, which gives rise to a Puiseux expansion of the form

$$z_0 = \sum_k \lambda^{k/n} p_k(z_1, \ldots, z_n); \tag{5}$$

thus, Brauner says, functions of multiple variables behave very similarly at Type I singularities as functions of one variable do.

The behavior of functions at Type II points, however, is more complicated and corresponds to branch points in the original function. As an illustrative example, Brauner gives z as a function of x and y defined implicitly by the equation $f(z, x, y) = z^3 - 3zx + 2y = 0$. This equation has elliptic discriminant $D(x, y) = x^3 - y^2$, which has

[2]This is bijective in sufficiently small neighborhoods because $\xi \approx z_1$.

a singularity of Type II at the origin.

Brauner is predominantly interested in studying the branching behavior of functions of two complex variables defined implicitly by a polynomial function; e.g. z defined implicitly by (x, y) as $f(z, x, y) = 0$ where f is a polynomial function. At non-singular points of f, Puiseux expansions for z may be computed easily, but this fails where f has a singularity. Brauner exhibits a connection between the behavior of such implicit functions and the behavior of the resulting manifold of zero values of the corresponding elliptic discriminant.

We forgo a detailed summary of Brauner's cumbersome methods; in two variables, his basic method is to take an intersection of the zero locus $D^{-1}(0)$ of the discriminant with a small sphere of three real dimensions and radius ϵ. The resulting curve has one real dimension, and Brauner shows that the curve is knotted if and only if the underlying branch point is of Type II. Brauner's methods were later modified by Kähler [9], who substituted the sphere with a different shape called the *polydisk* and in so doing sidestepped many algebraic complications introduced by Brauner's use of the stereographic projection of the 3-sphere.

3 Construction of the Milnor fibration

Our exposition in this section and the following is taken, by and large, from Milnor [14], Joubert [8], and Martinez [13].

Remark. We use the following notational conventions throughout the paper: $f\colon \mathbf{C}^m \to \mathbf{C}$ is a function of m complex variables, $n = m-1$, and the variables of f are numbered $z_0, \ldots, z_n$.

The embedding of a knot $K \subset S_\epsilon^{2n+1} \subset \mathbf{C}^2$ constructed from a singularity of a function $f\colon \mathbf{C}^m \to \mathbf{C}$ by the above construction of Brauner and Kähler gives some information about the topology of the singularity. Much information about the knot can be gleaned from studying its complement $S_\epsilon^{2n+1} \setminus K$, in particular the fiber of a particular fibration over the circle that assigns to each point $z \in S_\epsilon^{2n+1}$ the complex argument of $f(z)$.

Definition. Given polynomials $f_1, \ldots, f_r$ of m complex variables, let $V(f_1, \ldots, f_r)$ denote the subset of $\mathbf{C}^m$ on which all of these polynomials vanish. Such a set is called an *algebraic set* or *variety*, and corresponds to an ideal $I(V)$ in $\mathbf{C}[x_1, \ldots, x_m]$.

Definition. A topological space X can be given the structure of a cone over $Y \subset X$ if some point $x_0 \notin Y$ can be selected and non-intersecting lines drawn from x_0 to every point in Y such that the union of all the lines is Y.

Proposition 1 (Milnor). *For any sufficiently small sphere S_ϵ^{2n+1} centered at a non-singular or isolated singular point x of an algebraic set V, the intersection $S_\epsilon^{2n+1} \cap V$ is transverse and thus creates a (possibly empty) real manifold.*

Proof. See [14] 2.9, p. 17. □

We now show that if we choose spherical neighborhoods around regular points or isolated singularities, then all topological information about the intersection of the neighborhood and the algebraic set is in a sense encoded in the ball's surface.

Definition. The *cone* over a topological space X is the space $X \times [0,1]$ with all points $(x, 0)$ identified. In particular, if B^n is the solid ball of dimension n, then $B^n \cong C(S^{n-1})$, where the fibers $\{x\} \times [0,1]$ are radial line segments from the ball's center to the surface and the point $(x, 0)$ that the fibers all have in common is the center.

Lemma 2 (Milnor). *Let V be an algebraic set, and let x be either non-singular or an isolated singular point, let $B_\epsilon(x)$ be a neighborhood of sufficiently small radius ϵ about x, and let $S_\epsilon(x)$ be the surface of this neighborhood. Then for all sufficiently small radii ϵ, $(B_\epsilon(x), B_\epsilon(x) \cap V) \cong (C(S_\epsilon(x), C(S_\epsilon(x) \cap V)))$.*

Proof. We follow [8] 2.3.4 or [14] 2.10.

We may assume that ϵ is small enough, by Proposition 1, that the intersection of $S_{\epsilon'}(x)$ with V is transverse, and $B_{\epsilon'}(x)$ and contains no singular points of V besides x itself, for all $\epsilon' < \epsilon$. Define the radius function $r(y) = |y - x|^2$. For any point $y \in S_{\epsilon'}$, $\ker dr(y) = T_y S_{\epsilon'}$; as V and $S_{\epsilon'}$ intersect transversally, some vector in $T_y V$ cannot be in $T_y S_{\epsilon'}$, so any point in $V \cap S_{\epsilon'}$ must be a regular point of $r|_V$.

The idea of the proof is to draw lines from x to the surface $S_\epsilon(x)$ whose union is $B_\epsilon(x)$, and that either stay in V or are disjoint from V except for x. These lines are the solution curves of a particular vector field, so by standard results on uniqueness of solutions to vector fields, they do not intersect each other. By construction, this vector field will have the following two desirable properties:

1. It points away from x, so each of its solution curves intersects any sphere around x at most once.

2. At points in V other than the singularity x, it is tangent to V, so solution curves either lie entirely in V or have no points other than x in V.

When this is done, the same correspondence of lines in $B_\epsilon(x)$ to fibers of $C(S_\epsilon(x))$ will also exhibit $B_\epsilon(x) \cap V \cong C(S_\epsilon(x) \cap V)$. The vector field v_α is defined locally at every point $y_\alpha \in B_\epsilon \setminus \{x\}$ as follows:

1. If $y_\alpha \notin V$, then choose a neighborhood $U_\alpha \ni y_\alpha$ that does not intersect v and let $v_\alpha(y) = y - x$.

2. If $y_\alpha \in V$ (and is necessarily nonsingular), then choose local coordinates $u_1, \ldots, u_n$ in a neighborhood U_α of about y_α such that V corresponds to $u_1 = \cdots = u_k = 0$.[3]

[3]This is guaranteed by the local-immersion theorem in differential topology (see [6], p. 15).

Then v is given by varying the other coordinates $u_{k+1}, \ldots, u_n$; the tangent vectors $\frac{\partial}{\partial u_{k+1}}(y_\alpha), \ldots, \frac{\partial}{\partial u_n}(y_\alpha)$ span the tangent space of V, and as y_α is (as noted) a regular point of r, one of the the quantities $\frac{\partial r}{\partial u_h}$ for $k+1 \leq h \leq n$ must be nonzero. Define $v_\alpha(y) = \pm\frac{\partial}{\partial u_h}(y) = \pm(\frac{\partial y_1}{\partial u_h}, \ldots, \frac{\partial y_n}{\partial u_h})$, choosing the plus or minus sign in accordance with the sign of $\frac{\partial r}{\partial u_h}$.

These local definitions may be combined into a global definition $v = \sum_\alpha \lambda_\alpha v_\alpha$ by the use of a partition of unity $\{\lambda_\alpha\}$ of S_ϵ subordinate to the neighborhoods U_α used previously. See [8], p. 4 for the verification that this definition satisfies the two desired criteria, and for other technical details. $\square$

Henceforth, write simply S_ϵ^{2n+1} instead of $S_\epsilon^{2n+1}(x)$; this will be disambiguated if necessary. Further assume that ϵ is sufficiently small that $V \pitchfork S_\epsilon^{2n+1}$, as per Proposition 1. Let $K \cong V \cap S_\epsilon^{2n+1}$, and identify S^1 with the unit circle in the complex plane. Let $\phi\colon S_\epsilon \setminus K \to S^1$ be given as $\phi(z) = \frac{f(z)}{|f(z)|}$. Further make the *uncommon* definition that the gradient of a function $g(z_0, \ldots, z_n)$ is

$$\operatorname{grad} g = \left(\frac{\partial \overline{g}}{\partial z_0}, \ldots, \frac{\partial \overline{g}}{\partial z_n}\right); \tag{6}$$

this is the *complex conjugate* of the usual definition, with the advantage that the expression for the directional derivative of g becomes simply

$$D_v g(z) = \langle v, \operatorname{grad} z\rangle \tag{7}$$

with $\langle a, b\rangle = \sum_i a_i \overline{b_i}$ the usual Hermitian inner product (see [8], p. 10). The usual expression for the logarithmic derivative becomes

$$\operatorname{grad} \log f(z) = \frac{\operatorname{grad} f(z)}{\overline{f(z)}}, \tag{8}$$

which does not depend on which branch of the logarithm is chosen. Note, furthermore, that $\operatorname{grad}(\lambda g) = \overline{\lambda} \operatorname{grad} g$.

Remark. A complex vector space $\mathbf{C}^m$ may be regarded as a real vector space $\mathbf{R}^{2m}$, with real inner product given simply by the real part of the Hermitian inner product.

Lemma 3 (Milnor). *The map ϕ has a critical point at $z \in S_\epsilon^{2n+1} \setminus K$ if and only if $i \operatorname{grad} \log f(z)$ is a real multiple of z.*

Proof. See [14] 4.1, pp. 33–35 or [8] 2.5.1, pp. 10–11. Let $\theta(z)$ be the multiple-valued function such that $e^{i\theta(z)} = f(z)/|f(z)|$; θ has branches differing by 2π and a well-defined derivative. Taking the logarithm of $e^{i\theta(z)} = f(z)/|f(z)|$ gives $i\theta(z) = \log f(z) - \log|f(z)|$; the last term on the right is real. Multiply by $-i$ and take real parts to get

$\theta(z) = \Re(-i \log f(z))$. Given a curve $z = p(t)$, the derivative of $\theta(p(t))$ with respect to t is given by

$$\frac{d\theta(p(t))}{dt} = \Re \frac{d}{dt}\left((-i \log f(p(t))\right) \tag{9}$$

$$= \Re \left\langle \frac{dp}{dt}, \operatorname{grad}(-i \log f) \right\rangle \tag{10}$$

$$= \Re \left\langle \frac{dp}{dt}, i \operatorname{grad} \log f \right\rangle . \tag{11}$$

This is the expression for the directional derivative in the direction $v = p'(t)$:

$$D_v\theta(z) = \Re \left\langle v, i \operatorname{grad} \log f(z) \right\rangle , \tag{12}$$

where v is tangent to a sphere about the origin at a point z if and only if $\Re \langle v, z\rangle = 0$. Therefore, if $i \operatorname{grad} \log f(z)$ is a real multiple of z, then it is also normal to any tangent vector v; the directional derivatives $\Re \langle v, i \operatorname{grad} \log f(z)\rangle$ are zero, so θ and therefore ϕ have critical points on the sphere. Conversely, if $i \operatorname{grad} \log f(z)$ and z are linearly independent, then there is some vector v tangent to S^{2n+1}_ϵ such that $\Re \langle v, z\rangle = 0$ but $\Re \langle v, i \operatorname{grad} \log f(z)\rangle \neq 0$. Hence, θ and thus ϕ have nonzero directional derivatives in the direction of v, and thus do not have critical points. □

Remark. Lemma 6 strengthens this result.

Lemma 4 (Milnor's Curve-Selection Lemma)**.** *Let $V \subset \mathbf{R}^r$ be a real algebraic set, let $g_1, \ldots, g_s$ be polynomials, and define*

$$U = \{x \in \mathbf{R}^r | g_1(x) > 0, \ldots, g_s(x) > 0\} . \tag{13}$$

If $0 \in \overline{U \cap V}$, then there is some smooth curve $\gamma\colon [0, \delta) \to \mathbf{R}^r$ such that $\gamma(0) = 0$ and $\gamma(t) \in U \cap V$.

Proof. See [8] 2.4.1, p. 7 or [14] 3.1, p. 25. □

Remark. This result is a useful building block of several future results because it often allows the existence of a smooth curve from the origin containing only points of a certain type to be inferred from the mere existence of such points arbitrarily close to the origin.

Lemma 5 (Milnor)**.** *If $p\colon [0, \epsilon) \to \mathbf{C}^m$ is a smooth path, $p(0) = 0$, and $p(t) \in \mathbf{C}^n \setminus V$ for $t > 0$, and if furthermore $\operatorname{grad} \log f(p(t)) = \lambda(t)p(t)$ where $\lambda(t) \in \mathbf{C}$, then $\lambda(t) \neq 0$ for small t, and $\lim_{t\to 0} \arg \lambda(t) = 0$.*

Proof. See [8] 2.5.3, p. 12. □

Lemma 6 (Milnor). *There is some $\epsilon > 0$ such that for $z \in \mathbf{C}^m \setminus V$ and $|z| \le \epsilon$, either z and $\operatorname{grad}\log f(z)$ are linearly independent over $\mathbf{C}$, or $\operatorname{grad}\log f(z) = \lambda z$, where λ is nonzero and $|\arg\lambda| < \pi/4$.*

Proof. See [8] 2.5.2, p. 12. The essential method of proof is to suppose that there are points z arbitrarily close to the origin that fail to satisfy the requirements of this lemma, and use Lemma 4 to construct a curve out of them. Lemma 5 gives a contradiction. □

Remark. Note that $|\arg\lambda| < \pi/4$ implies $\Re(\lambda) > 0$. This will be used in Lemma 11 and Proposition 25.

The upshot of these results is that for all ϵ sufficiently small, the map $\phi\colon S^{2n+1}_\epsilon \setminus K \to S^1$ has no critical points. The preimages $\phi^{-1}(e^{i\theta})$ of regular values of a map to one-dimensional manifold S^1 have codimension 1; therefore, each of these preimages is a manifold of codimension 1 and dimension $2n$.

4 Alternate construction of the Milnor fibration

A second construction of the Milnor fibration is given in [8] 2.6. We omit a detailed exposition of this construction, but give an overview of the key definitions and results.

Given a complex polynomial f with an isolated singularity at 0, we know from Lemma 1 that $f^{-1}(0)$ and S^{2n+1}_ϵ intersect transversally. It may in fact further be shown:

Lemma 7. *Let $D_\delta = \{z \in \mathbf{C} | |z| < \delta\}$, and let $f\colon \mathbf{C}^m \to \mathbf{C}$ have a regular isolated singular point at the origin. Let ϵ be chosen sufficiently small, according to Lemma 1, that $f^{-1}(0) \pitchfork S^{2n+1}_\epsilon$.*

Proof. See [8] 2.6.1, p. 17. □

This result allows the application of Ehresmann's fibration theorem:

Proposition 8 (Ehresmann). *If $f\colon E \to B$ is a proper[4] submersion of the manifold E into B, then it is a locally trivial fibration (meaning that around neighborhoods of every $b \in B$ and $f^{-1}(b)$, E looks like the product space of B with some other space, and f like the projection map). More precisely, given any $b \in B$, we can choose a neighborhood $U \ni b$ and a diffeomorphism $t\colon f^{-1}(U) \to U \times f^{-1}(b)$ such that $\pi \circ t = f$, where π is the projection map from $U \times f^{-1}(b) \to U$. Moreover, if $A \subset E$ is closed and $f|_A$ is also a submersion, then we can choose t to map $f^{-1}(U) \cap A$ to $U \times (f^{-1}(b) \cap A)$, thus exhibiting A as locally a product space over B, and call f a locally trivial fibration of (E, A) over B.*

Proof. See Proposition 2 of [4]. □

[4] A map is *proper* if every compact subset of its codomain has a compact preimage.

Proposition 9. *Let $\epsilon, \delta > 0$ be such that $f^{-1}(t)$ intersects S_ϵ^{2n+1} transversally wherever $t \in D_\delta$. Let $B = D_\delta \setminus \{0\}$ be a puncturing of D_δ, define $E = \overline{B_\epsilon} \cap f^{-1}(D_\delta)$, and let $\tilde{\psi}$ be the restriction of ψ to E. Then $\tilde{\psi}$ is a locally trivial fibration of (E, A) over B, in the sense shown above.*

Proof. See [8] 2.6.3, p. 18. We must show that f is a proper submersion both of E and of ∂E; Ehresmann's fibration theorem gets us the rest of the way. To see that $\tilde{\psi}$ is proper, note that for any compact (and thus) closed set $V \subset B$, $\tilde{\psi}^{-1}(V)$ is both closed (because it is the continuous preimage of a closed set) and bounded (because the enclosing space $\overline{B_\epsilon}$ is bounded), so it is compact. Thus, $\tilde{\psi}$ is proper. To show that $\tilde{\psi}$ is a submersion, take an interior point x of E; then $\tilde{\psi} = f$ in a neighborhood of x, and f is a submersion of E into B away from the origin, so $\tilde{\psi}$ is also a submersion of E. To show that f and thus $\tilde{\psi}$ are still a submersion on ∂E, take $x \in \partial E$ and let $y = f(x)$. By Lemma 7, the intersection of $f^{-1}(y)$ and S_ϵ at x is transverse, so $f|_{\partial E}$ is a submersion. $\square$

One may show further that if δ is sufficiently small, then shrinking δ further gives an equivalent fibration of D_δ. The Milnor fibration is obtained by restricting $\tilde{\psi}$ to the preimage of a circle $\partial D_{\delta'}$, where $\delta' < \delta$. It may be shown (see [8] 2.6.5 and 2.6.6, pp. 20–22) that this fibration on $S_\epsilon^{2n+1} \setminus f^{-1}(0)$ is equivalent to the fibration ϕ on $S_\epsilon^{2n+1} \setminus f^{-1}(\overline{D_\delta})$, in the sense that there exists a fiber-preserving diffeomorphism between them.

Remark. Milnor proves ([14], p. 53) that that each fiber $F_\theta = \phi^{-1}(e^{i\theta})$ is diffeomorphic to $f^{-1}(c)$, where c is some sufficiently small complex number and $\arg c = \theta$.

5 Morse theory on the Milnor fiber

Still identifying $S^1 \cong \{z \in \mathbf{C} | |z| = 1\}$ and letting $\phi \colon S_\epsilon^{2n+1} \setminus K \to S^1$ be the Milnor fibration, let $F_\theta = \phi^{-1}(e^{i\theta})$ denote a particular fiber. Define $a \colon S_\epsilon^{2n+1} \setminus K \to \mathbf{R}$ as $a(z) = \log |f(z)|$, and define $a_\theta = a|_{F_\theta}$. It may be assumed by perturbing a slightly (see [8], pp. 27–8) that a is a Morse function, i.e. that at no critical point of a the Hessian matrix of second derivatives of a vanish. (The proof is an application of Sard's Lemma from differential topology.)

Lemma 10 (Milnor)**.** *There is a critical point of a_0 at $z \in F_\theta$ if and only if* $\operatorname{grad} \log f(z)$ *is a complex multiple of z.*

Proof. The proof of this proposition, which may be found in [14] 5.3, p. 46, resembles that of Lemma 3. In any direction v,

$$D_v \log |f(z)| = D_v \Re \left(\log f(z)\right) \tag{14}$$
$$= \Re \langle v, \operatorname{grad} \log f(z) \rangle. \tag{15}$$

The point z is a critical point of F_θ if and only if this quantity is zero for all tangent vectors v of F_θ at z; this holds, in turn, if and only if $\operatorname{grad}\log f(z)$ is normal to F_θ at z.

By Lemma 3, the normal vectors of F_θ are spanned by z and $i\operatorname{grad}\log f(z)$ over $\mathbf{R}$; therefore, $\operatorname{grad}\log f(z) \perp F_\theta$ if and only if $\{z, \operatorname{grad}\log f(z), i\operatorname{grad}\log f(z)\}$ is not linearly independent over $\mathbf{R}$. In this case, we can write z as a complex multiple of $\operatorname{grad}\log f(z)$, and we are done. $\square$

We now seek to estimate the Morse index of a_θ at the singular point z, which is defined as the number of negative eigenvalues of its Hessian matrix. Given a curve $p\colon \mathbf{R} \to F_\theta$ such that $p(0) = z$ and $p'(0) = \vec{v}$, we may write

$$\frac{d^2 a_\theta(p(0))}{dt^2} = H(\vec{v}), \tag{16}$$

where H is some quadratic function of $\vec{v}$; we interpret H as the Hessian.

Remark. The Hessian matrix is symmetric and thus diagonalizable. The tangent space $T_z(F_\theta)$ may therefore be decomposed into the direct sum of the space spanned by the eigenvectors of H with negative eigenvalue, on which H is negative definite, and the the space spanned by the eigenvectors of H with zero or positive eigenvalue, on which H is positive semi-definite. This fact becomes important in our proof of Lemma 12

Lemma 11 (Milnor). *The following differential equation holds:*

$$\frac{d^2 a_\theta(p(0))}{dt^2} = \sum_{i,j=0}^{n} \Re(b_{ij} v_i v_j) - c|\vec{v}|^2, \tag{17}$$

where (b_{ij}) is a matrix of complex numbers and $c > 0$ is real.

Proof. The proof comes from [14] 5.5, pp. 47–48. Observe that F_θ is the set of values for which $f/|f| = e^{i\theta}$; this is a constant, so $a_\theta(p(t)) = \log f(p(t)) - i\theta$ may be differentiated and the constant $i\theta$ term eliminated to yield

$$\frac{da_\theta}{dt} = \frac{d}{dt}\log f = \sum_{i=0}^{n} \frac{\partial \log f}{\partial z_i}\frac{dp_i}{dt} \tag{18}$$

by the chain rule, where $p_i(t)$ is the i-th component of $p(t)$. Another differentiation gives

$$\frac{d^2 a_\theta}{dt^2} = \sum_{i=0}^{n} \frac{\partial \log f}{\partial z_i}\frac{d^2 p_i}{dt^2} + \sum_{i,j=0}^{n} \frac{\partial^2 \log f}{\partial z_i \partial z_j}\frac{dp_i}{dt}\frac{dp_j}{dt}. \tag{19}$$

If $t = 0$ and z is a critical point of f, then $\operatorname{grad}\log f(z) = \lambda z$ for some $\lambda \in \mathbf{C}$ (by

Lemma 6, λ has positive real part), and we may rewrite Equation 19 as follows:

$$\frac{\partial^2 a_\theta}{\partial t^2} = \left\langle \frac{d^2 p}{dt^2}, \lambda z \right\rangle + \sum_{i,j=0}^{n} D_{ij} v_i v_j. \tag{20}$$

Each of the two terms of this equation corresponds to a term in Equation 19, and we have introduced the notation $D_{ij} = \frac{\partial^2 \log f}{\partial z_i \partial z_j}$. a_θ and thus $\frac{\partial^2 a_\theta}{\partial t^2}$ are real-valued by definition. If both sides are multiplied by λ and the real part taken, then recalling that $\langle a, \lambda b\rangle = \overline{\lambda}\langle a, b\rangle$, we have

$$\frac{\partial^2 a_\theta}{\partial t^2} \Re(\lambda) = |\lambda|^2 \Re \left\langle \frac{d^2 p}{dt^2}, z \right\rangle + \sum_{i,j=0}^{n} \Re\left(\lambda D_{ij} v_i v_j\right). \tag{21}$$

As p is a curve on a sphere, $\langle p(t), p(t)\rangle = |p(t)|^2$ is a constant. Taking derivatives with the dot-product identity for vector-valued functions, and introducing the dot notation $\dot{p}(t) = \frac{dp(t)}{dt}$, we have

$$\begin{aligned}
0 &= \frac{d}{dt}\langle p(t), p(t)\rangle && (22)\\
&= \langle p(t), \dot{p}(t)\rangle + \langle \dot{p}(t), p(t)\rangle && (23)\\
0 &= \frac{d^2}{dt^2}\langle p(t), p(t)\rangle && (24)\\
&= \langle p(t), \ddot{p}(t)\rangle + \langle p(t), \ddot{p}(t)\rangle + 2\langle \dot{p}(t), \dot{p}(t)\rangle && (25)\\
&= 2\Re\langle p(t), \ddot{p}(t)\rangle + 2|\dot{p}(t)|^2. && (26)
\end{aligned}$$

Evaulated at $t = 0$, $\dot{p}(t) = \vec{v}$ and $p(t) = z$, so this gets us the identity $\Re\langle \ddot{p}(0), z\rangle = -|v|^2$. Substituting this into Equation 21 yields

$$\frac{\partial^2 a_\theta}{\partial t^2} \Re(\lambda) = -|\lambda|^2 |\vec{v}|^2 + \sum_{i,j=0}^{n} \Re(\lambda D_{ij} v_i v_j). \tag{27}$$

Divide through by $\Re(\lambda)$, which is positive by Lemma 6, and we get the desired result with $c = |\lambda|^2$ and $b_{ij} = H_{ij}$.

$\square$

This sets up the following result:

Lemma 12 (Milnor). *The Morse indices of $a_\theta \colon F_\theta \to \mathbf{R}$ and of $a \colon S_\epsilon^{2n+1} \to \mathbf{R}$ are both at least n.*

Proof. See [14] 5.6, p. 49. The quadratic function $H(\vec{v}) = \Re(\sum_{i,j=0}^{n} b_{ij} v_i v_j) - c|\vec{v}|^2$ defined previously has Morse index equal to the dimension of the largest linear subspace of its domain—in this case, the tangent space $T_z F_\theta$, treated as a vector space over $\mathbf{R}$—on which it is negative definite.

First note that if $H(\vec{v}) \geq 0$, then $H(i\vec{v}) < 0$, as replacing $\vec{v}$ with $i\vec{v}$ preserves $|\vec{v}|$ and hence the second term $-c|\vec{v}|^2 < 0$ of $H(\vec{v})$ while flipping the sign of the (necessarily positive) first term. Furthermore, T_zF_θ is in fact a complex vector space. If $v \in T_zF_\theta$, then $iv \in T_zF_\theta$ as well.

Decompose T_zF_θ as direct sum of real vector spaces $T_0 \oplus_{\mathbf{R}} T_1$ of total dimension $2n$, where H is negative definite on T_0 and positive semi-definite on T_1. By definition the Morse index $I = \dim_{\mathbf{R}} T_0$, but by our remark in the preceding paragraph, $I \geq \dim_R(iT_1) = \dim_R T_1$. Adding these equations, $2I \geq \dim_R T_0 + \dim_R T_1 = 2n$, so $I \geq n$. This establishes the claim about a_θ.

The Morse index of a_θ provides a non-strict lower bound on the Morse index of a, because $T_zF_\theta \subset T_zS_\epsilon^{2n+1}$ and any subspace of T_zF_θ on which H is negative definite is also a subspace of $T_zS_\epsilon^{2n+1}$. $\square$

Lemma 13 (Milnor)**.** *The critical points of a_θ (resp. a) are contained in the compact set $\{z \in F_\theta \,|\, |f(z)| \geq \eta_\theta\}$ for some η_θ (resp. $|f(z)| \geq \eta$ for some η).*

Proof. See [14] 5.7, pp. 49–50. Suppose to the contrary that there exists a sequence of critical points $\xi_0, \xi_1, \ldots$ of a_θ such that $\lim_{n\to\infty} f(\xi_n) = 0$. Then the curve selection lemma guarantees the existence of a path $p\colon (0, \epsilon) \to F_\theta$ consisting entirely of critical points, and $\lim_{t\to 0} p(t) = \xi_0$. Moreover, a_θ is constant on a path of its own critical points, so $|f|$ is also constant. But $|f|$ must tend to $|f(\xi_0)| = 0$ along this path, a contradiction. $\square$

Proposition 14 (Milnor)**.** *There is a smooth map $s_\theta\colon F_\theta \to \mathbf{R}_+$ such that all critical points of s_θ are nondegenerate with Morse index at least n. Similarly, there exists a map $s\colon S_\epsilon^{2n+1} \backslash K \to \mathbf{R}_+$ satisfying this same property, and such that $s(z) = |f(z)|$ when $|f(z)|$ is sufficiently small.*

Proof. We do not present this proof, which may be found at [14] 5.8, pp. 50–51. The basic idea of this proof (for which Milnor refers to Morse's work) is that we may define $s = |f|$ and then perturb s on a compact neighborhood of the degenerate critical points so as to eliminate their degeneracy while preserving the Morse index, and all points of $|f|$ have Morse indices at least n. $\square$

The results of these investigations are the following:

Proposition 15 (Milnor)**.** *The fibers F_θ are parallelizable, meaning that $2n$ real-valued vector fields may be defined on them whose derivatives at every point provide a tangent space, and they have the homotopy type of a finite CW-complex of dimension n.*

Proof. This is a consequence of the main theorem of Morse theory, and follows from the existence of a function $g(z) = -\log s_\theta(z)\colon F_\theta \to \mathbf{R}$ with no degenerate points and such that for every real number c, $\{z|g(z) \leq c\}$ is compact. The main theorem of Morse theory identifies every critical point of g (which has index I at most n) with a cell of dimension I in a CW complex. See [14] 5.1, p. 51 or [13] 7.1, p. 55 for detail. $\square$

Proposition 16 (Milnor)**.** *The space K is* $(n-2)$*-connected.*

Proof. See [14] 5.2, p. 51–52. □

Remark. There is in fact a more precise description of the homology of the fiber, as noted in [14], p. 52: if $N_\eta(K) \subset S_\epsilon^{2n+1}$ is the neighborhood of K containing all $z \in S_\epsilon^{2n+1}$ such that $|f(z)| < \eta$, then by Lemma 13, N_η may be made a smooth manifold with boundary by choosing η sufficiently small that the singular points of a lie outside $N_\eta(K)$. S_ϵ can then be built from $N_\eta(K)$ by adjoining handles of dimension at least n. This fact will be useful in proving Corollary 18.

6 Isolated critical points

The previous results can be strengthened under additional assumption that on some neighborhood of the origin in $\mathbf{C}^m$, $f\colon \mathbf{C}^m \to \mathbf{C}$ has no singular points except, possibly, at the origin itself. In this case, the regular part of $f^{-1}(0)$ in a neighborhood of the origin is a *Stein manifold* (see [12], p. 3). The principal results are the following:

Proposition 17 (Milnor 6.1)**.** *With the radius* ϵ *of the intersecting sphere taken sufficiently small, the closure of each fiber* F_θ *in* S_ϵ^{2n+1} *is a* $2n$*-dimensional manifold with boundary* K*.*

Proof. We prove this for the fiber F_0; similar arguments hold for the other fibers F_θ, $\theta \neq 0$. It may be shown that $f\colon S_\epsilon^{2n+1} \to \mathbf{C}$ has no critical points on K if ϵ is sufficiently small; this folows either from Lemma 4 (the Curve-Selection Lemma) or from Milnor's proof of Proposition 1. We give a proof from the Curve-Selection Lemma. The critical points of $f|_{S_\epsilon^{2n+1}}$ are those for which $\operatorname{grad} f(z)$ is a complex multiple of z and thus normal to S_ϵ^{2n+1}. If such critical points may be found for ϵ arbitrarily small, then for some ϵ, there exists a non-constant curve $p\colon [0,\epsilon) \to \mathbf{C}^m$ consisting of such points, with $p(0) = 0$ and $f(p(t)) = 0$, then $\langle \dot{p}, \operatorname{grad} f \rangle = \frac{d}{dt} f(p(t)) = 0$. In particular,

$$2\Re \langle \dot{p}(t), p(t) \rangle = \frac{d}{dt} |p(t)|^2 = 0; \tag{28}$$

thus $p(t)$ must be a constant zero, a contradiction.

For ϵ sufficiently small, take $z_0 \in K$ and choose real local coordinates $u_1, \ldots, u_{2n+1}$ on a neighborhood $U \ni z_0$ such that $f(z) = u_1(z) + iu_2(z)$.[5] A point of U belongs to $F_0 = \phi^{-1}(1) = f^{-1}(\mathbf{R}_+)$ if and only if $u_1 > 0$ and $u_2 = 0$, so $\overline{F_0} \cap U$ is the set given by $u_1 \geq 0, u_2 = 0$, and $u_3, \ldots, u_{2n+1}$ free to assume any value.[6] This is a smooth $2n$-dimensional manifold with boundary given by $u_1 = u_2 = 0$, which corresponds to $f(z) = 0$ and thus $z \in K \cap U$.

□

[5]These will approximate corresponding vectors v_1, v_2 such that $D_{v_1} f(z_0) = 1$ and $D_{v_2} f(z_0) = i$.

[6]The criteria on u_1 and u_2 must be changed for a fiber F_θ, $\theta \neq 0$; the rest of the argument still holds.

Remark. Martinez ([12], p. 2) remarks that another way of putting the above result is that the Milnor fibration of an isolated singularity, becomes an *open-book decomposition* S_ϵ^{2n+1}, in which the "pages" are the sets F_θ whose closures are disjoint except at their shared "binding" K.

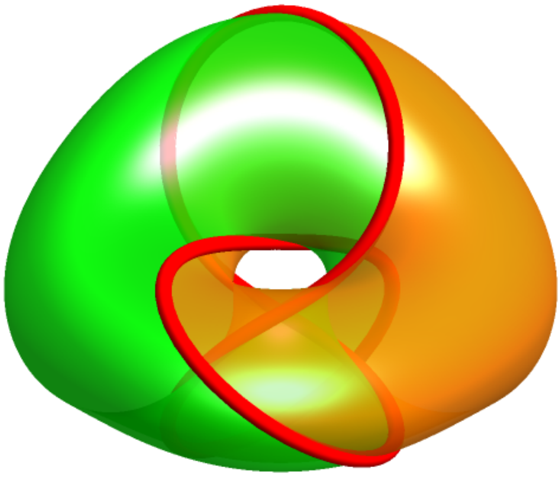

Figure 2: A trefoil knot with Milnor fiber (from [13]).

Corollary 18 (Milnor). *The closure $\overline{F}_\theta = F_\theta \cup K$ has the same homotopy type as the complement $S_\epsilon^{2m-1} \setminus \overline{F}_\theta$.*

Proof. See [14] 6.2, p. 56. $S_\epsilon^{2m-1} \setminus \overline{F}_\theta$ contains all the fibers $F_{\theta'}$ for $\theta' \neq \theta$ and is therefore a locally trivial fibration over the contractible space $S^1 \setminus \{e^{i\theta}\} \cong [0,1]$. This space can be retracted to a single fiber $F_{\theta'} \cong F_\theta$ while preserving homotopy.[7] $\square$

Corollary 19 (Milnor). *The fiber F_θ has the homology of a point in fewer than n dimensions: $H_0 \cong \mathbf{Z}$ and $H_i = 0$ for $1 \leq i < n$.*

Proof. See [14] 6.3, p. 57. This is a consequence of the Alexander duality theorem: $\tilde{H}_i(S_\epsilon \setminus \overline{F}_\theta) = \tilde{H}^{2n-i}(\overline{F}_\theta)$, where $\tilde{H}_i$ and $\tilde{H}^i$ are respectively the reduced homology and reduced cohomology groups.[8] $\square$

Lemma 20 (Milnor). *The space F_θ is $(n-1)$-connected.*

Proof. See [14] 6.4, p. 57. All reduced homology groups $\tilde{H}_i(F_\theta)$ vanish for $i \leq n$, so we must only prove that F_θ is simply connected as long as $n \geq 2$.

[7] It is a general fact that if $\pi\colon E \to B$ is a fibration with representative fiber $F = \pi^{-1}(b)$ for some $b \in B$ is a contractible space, then E and F have the same homotopy groups.

[8] The reduced homology group $\tilde{H}_i(X)$ is the kernel of the natural homomorphism $H_i(X) \to H_i(\text{point})$, and the reduced cohomology group $\tilde{H}^i(X)$ is the cokernel of the map $H^i(\text{point}) \to H^i(X)$. Very intuitively speaking, these are the homology groups readjusted so that the point has vanishing homology groups of all orders. In the case of homology groups, $H_i(X) = \tilde{H}_i(X)$ except in the case $i = 0$, where $\operatorname{rank} \tilde{H}_0(X) = \operatorname{rank} H_0(X) - 1$.

By logic similar to our remark on Proposition 16, $\overline{F_\theta}$ can be built up by starting with a disk D_0^{2n} of $2n$ dimensions and adjoining handles contained within S_ϵ^{2n+1} of dimension at most n. (The full explanation requires the use of $-s_\theta$ as a Morse function.) A sphere minus a disk is simply connected, and adding handles to the disk of dimension less than $\dim(S_\epsilon^{2n+1}) - 3 = 2n-2$, does not change the fundamental group of its complement. It thus follows that, provided $n \leq 2n-2$ (i.e. $n \geq 2$), $S_\epsilon^{2n+1} \setminus \overline{F}_\theta$ is simply connected. By Corollary 18, $\overline{F}_\theta$ is simply connected as well, and the conclusion follows. □

Definition. The wedge sum of a set of spaces $X_1, \ldots, X_n$ with distinguished points $x_1, \ldots, x_n$ is the union $X_1 \cup \ldots \cup X_n$, modulo an equivalence relation that identifies the all the points x_i.

Proposition 21 (Milnor)**.** *Each fiber has the homotopy type of a wedge sum $S^n \vee \ldots \vee S^n$.*

Proof. See [14] 6.5, p. 57–58. Each homology group $H_n(F_\theta)$ must be free abelian. If it were not, then a torsion element would produce a cohomology class of dimension $n+1$. This contradicts Proposition 15, under which F_θ must be homotopic to a structure of dimension n. According to a theorem of Hurewicz, a free abelian homology group has a basis of finitely maps $S^n \to F_\theta$ that take a fixed base point in S^n to a fixed base point in F_θ. These may therefore be combined into a map $S^n \vee \cdots \vee S^n \to F_\theta$. Furthermore, this map, by a theorem of Serre (see [13] 1.11, p. 55), gives an isomorphism of the n-th (and only nontrivial) homology group.

□

Most homology groups of the fiber can be computed from this result and from the general result that for a set of manifolds $X_\alpha, \alpha \in A$,

$$\tilde{H}_n\left(\bigvee_{\alpha \in A} X_\alpha\right) = \bigoplus_{\alpha \in A} \tilde{H}_n(X_\alpha). \tag{29}$$

(This statement is true of ordinary as well as reduced homology groups for $n \neq 0$.) As all reduced homology groups of the sphere vanish except for $\tilde{H}_n(S^n) \cong \mathbf{Z}$, this proves:

Corollary 22. *The following isomorphisms hold:*

$$\tilde{H}_k(F_\theta) \cong \begin{cases} 0 & k \neq n \\ \mathbf{Z}^\mu & k = n, \end{cases} \tag{30}$$

for some integer μ.

Remark. One may find another derivation of Corollary 22 in [8], chp. 3, p. 32.

Definition. The integer $\mu = \operatorname{rank} \tilde{H}_k(F_\theta)$ is the *Milnor number*.

Remark. The Milnor number has various interpretations. Although it is defined as the rank of the middle homology group of the Milnor fiber, it also may be interpreted as a geometric multiplicity, as we shall see presently. There are also other interpretations of the Milnor number, both algebraic and topological in nature.

7 The Milnor number

Definition. Let $g\colon \mathbf{C}^m \to \mathbf{C}^m$ be an analytic function with an isolated zero z_0. The *multiplicity* μ of g is the topological degree of the map $(z \mapsto g(z)/|g(z)|)\colon S_\epsilon^{2m-1}(z_0) \to S_1^{2m+1}(0)$ for $\epsilon \geq 0$ small.

Remark. We shall show that this multiplicity μ equals the Milnor number defined previously.

Proposition 23 (Lefschetz, Milnor)**.** *The multiplicity μ is a positive integer.*

Proof. See [14] appendix B, pp. 110–115. □

Remark. Lefschetz ([11], p. 382) offers a different argument from the intersection theory of algebraic varieties. He uses a different but equivalent definition of the multiplicity of the intersection of two algebraic varieties to equal the number of non-singular intersection points of a slightly perturbed, transverse intersection.

We specialize to the case of a polynomial $f\colon \mathbf{C}^m \to \mathbf{C}$ with an isolated singularity at the origin. The degree μ, defined topologically previously, is also the multiplicity of $z = 0$ as an isolated solution to the system of polynomial equations.

$$\frac{\partial f(0)}{\partial z_1} = \cdots = \frac{\partial f(0)}{\partial z_m} = 0. \tag{31}$$

As before, let $n = m - 1$; the Milnor fibration of $S_\epsilon^{2n+1} \setminus f^{-1}(0)$ over S^1 has fibers of the form

$$F_\theta = \left\{ z \in S_\epsilon^{2n+1} \setminus f^{-1}(0) \middle| \arg f(z) = \theta \right\}. \tag{32}$$

Each fiber is a real manifold of $2n$ dimensions; we will take F_0 (on which $f(z)$ is positive real) as the prototypical fiber.

Definition. The Betti numbers of a real manifold of dimension k, numbered $b_0, \ldots, b_k$, are the ranks of the homology groups and give the numbers of "holes" of various dimensions in the manifold; thus is homology is defined over $\mathbf{Z}$ (which is the usual convention that makes H_1 correspond to the fundamental group), then $H_i \cong \mathbf{Z}^{b_i}$. If k is even, then the "middle" Betti number is $b_{k/2}$.

Another preparatory lemma, which gives us a more convenient way of computing the degree of the map of a sphere onto itself, is required for the main theorem.

Lemma 24 (Milnor). *Let S^k be the unit sphere of $\mathbf{R}^{k+1}$, and let $v\colon S^k \to S^k$ be smooth. Suppose further that there exists a compact set $M \subset S^k$, with inward normal vector $n(x)$ at every point $x \in \partial M$ (meaning the tangent vector to S^k that is perpendicular to ∂M and points into M), such that M and v satisfy the following conditions:*

1. *M contains every fixed point of v;*
2. *$v(x) \neq -x$ for all $x \in M$; and,*
3. *$\langle v(x), n(x)\rangle > 0$ for all $x \in \partial M$.*

Then the Euler characteristic χ of M and the degree d of v are related as $\chi = 1 + (-1)^k d$.

Proof. See [14] 7.4, pp. 61–62. This is a result in Lefschetz theory. We can assume that v has only isolated fixed points by adding a small perturbation to regions where v has non-isolated fixed points; this may be done without altering the map's homotopy properties (see [6], p. 120).

The *Lefschetz number* $L(x)$ of any fixed point x of a map $f\colon X \to X$, recall, is the degree of the map $z \mapsto \frac{f(z)-z}{|f(z)-z|}$. If f is a Lefschetz map (meaning that at no fixed point x does $df_x\colon T_xX \to T_xX$ have 1 as an eigenvalue), then it may also be defined as the sign of the determinant of $df_x - I\colon T_xX \to T_xX$; these two definitions[9] are equivalent (for proof, see [6], p. 128), and any map may be perturbed arbitrarily little to be made a Lefschetz map.

A theorem of Lefschetz states that the sum of each of these indices equals the Lefschetz number

$$\sum_{j\geq 0}(-1)^j \operatorname{tr}(v_*) = 1 + (-1)^k d \tag{33}$$

where $v_*\colon H_j(S^k) \to H_j(S^k)$ is the pushforward map induced by v on homology groups and all homology groups except $H_0(S^k)$ and $H_j(S^k)$ vanish. Now define for $0 \leq t \leq 1$ a family of maps $v_t\colon M \to S^k$ given by

$$v_t(x) = \frac{(1-t)x + tv(x)}{|(1-t)x + tv(x)|}. \tag{34}$$

As $v(x) \neq -x$ for any $x \in M$, this expression is always defined. Furthermore, the criterion $\langle v(x), n(x)\rangle > 0$ means that for t small, increasing t means moving boundary points inward. Therefore, v_t takes M into itself for $0 < t \leq \epsilon$, and the Lefschetz number of v_t equals the Euler number $\chi(M)$. The fixed points of v_t, however, are the same as those of v; as the Lefschetz index of a fixed point x is an integer and various continuously (i.e. cannot change) with t, the Lefschetz number $\chi(M)$ of v_ϵ equals the Lefschetz number $1 + (-1)^k d$ of $v = v_0$.

[9]The Lefschetz number is more properly defined as the intersection number $I(\Delta, \mathrm{graph}(f))$ as submanifolds of $X \times X$, where $\Delta \subset X \times X$ is the diagonal of X; see [6] for a proof of their equivalence.

□

The following is the key result of the mathematical portion of the paper; it establishes a link between the topology of the fiber and the algebra of the corresponding polynomial.

Proposition 25 (Milnor)**.** *The middle Betti number b_n and multiplicity μ are equal.*

Proof. See [14] 7.2, pp. 60, 62–64. Let $M = \{z \in S_\epsilon^{2n+1} | \Re(f(z)) \geq 0\} = K \cup \bigcup_{|\theta| \leq \pi/2} F_\theta$. M is a closed subset[10] of a compact set S_ϵ^{2n+1}, and is therefore compact. Each fiber has boundary K, so $\partial M = F_{-\pi/2} \cup K \cup F_{\pi/2}$. Furthermore, M is homotopic to F_θ by retracting the underlying interval $[-\pi/2, \pi/2]$ to a point.[11]

Define $v\colon S_\epsilon^{2n+1} \to S_\epsilon^{2n+1}$ as $v(z) = \epsilon \frac{\operatorname{grad} f(z)}{|\operatorname{grad} f(z)|}$. We claim that this satisfies the conditions of Lemma 24, which we check one by one.

1. There is a fixed point of v at z if and only if $\operatorname{grad} f(z)$ is a positive real multiple of z. Write $\operatorname{grad} f(z) = cz$. For ϵ sufficiently small, this can only be true if $f(z) \neq 0$, by reasoning similar to the proof of Proposition 17 showing that 0 is a regular value of f, and $\operatorname{grad} \log f(z) = \frac{c}{\overline{f(z)}} z$. We have previously shown in Lemma 6 that if z and $\operatorname{grad} \log f(z)$ are not linearly independent, then $\operatorname{grad} \log f(z) = \lambda z$ where $|\arg \lambda| < \frac{\pi}{4}$. In particular, $\lambda = \frac{c}{\overline{f(z)}}$ has positive real part; so, therefore, must $f(z)$, so $z \in M$ by definition of M.

2. This argument resembles the previous one. If $v(z) = -z$, then z is a fixed point of $-v = \epsilon \frac{\operatorname{grad}(-f)}{|\operatorname{grad}(-f)|}$, which holds if and only if $\operatorname{grad} f(z)$ is a *negative* real multiple of z. In this case, $f(z) \neq 0$, and $\operatorname{grad} \log f(z) = \frac{c}{\overline{f(z)}}$ where $c < 0$ and the coefficient $\frac{c}{\overline{f(z)}}$ must have positive real part. Thus, $\Re \overline{f}(z) < 0$, so $z \notin M$ by definition of M.

3. Take $z \in \partial M$, and let $p\colon I \to M$ (where $I \ni 0$ is a closed real interval) be a path such that $p(0) = z$ and $p(\epsilon) \in M$ for sufficiently small positive ϵ (that is, p crosses into M). $\Re(f(z)) = 0$ and $\Re(f(\epsilon)) > 0$, so
$$\frac{d\Re(f(p(t))}{dt} = \Re\langle \frac{dp}{dt}, \operatorname{grad} f \rangle > 0. \tag{35}$$
Therefore, $\Re\langle n(z), v(z) \rangle > 0$.

We may thus apply the result of Lemma 24, the formula $\chi(M) = 1 + (-1)^{2n+1} d$, where d is the degree of v. Here, $2n+1$ is odd, M can be retracted to F_θ, and retractions preserve Euler characteristics, so we may write instead $\chi(F_\theta) = 1 - d$.

[10] It is closed as the preimage of the closed set $\Re(z) \geq 0$ under the continuous map $z \mapsto \Re(f(z))$.

[11] We used the same property of fibrations over compact spaces in the proof of Corollary 18.

Furthermore, $\deg v = (-1)^m \mu = -(-1)^n \mu$, where μ is the multiplicity of the origin as the solution to the system of equations

$$\frac{\partial f}{\partial z_1} = \cdots = \frac{\partial f}{\partial z_m} = 0, \tag{36}$$

because μ is the degree of the map $z \mapsto \frac{g(z)}{|g(z)|}$, where $g(z) = \overline{\operatorname{grad} f(z)}$ (recall our uncommon convention for the meaning of grad) and the conjugation map, a reflection over each of the m coordinate axes, carries S_ϵ^{2n+1} into itself with degree $(-1)^m$. Therefore, the formula $\chi(F_\theta) = 1 - \deg v$ becomes $\chi(F_\theta) = 1 + (-1)^n \mu$. However, by definition,

$$\chi = \sum_j (-1)^j \operatorname{rank} H_j(F_\theta), \tag{37}$$

and all homology groups of F_θ except $H_0(F_\theta)$ and $H_n(F_\theta)$ vanish, so this gives

$$1 + (-1)^n \operatorname{rank} H_n(F_\theta) = 1 + (-1)^n \mu, \tag{38}$$

so

$$\mu = \operatorname{rank} H_n(F_\theta). \tag{39}$$

This completes the proof.

□

8 Other interpretations of the Milnor number

Several equivalent definitions of the Milnor number for a function f with corresponding prototypical Milnor fiber $F_{f,0}$ are listed in [13], p. 526; we list here, with a prefatory definition:

1. The *geometric index* is the local geometric multiplicity of f in a neighborhood of the origin. This is defined as a notion in Morse theory, as the number of Morse points into which the origin splits if a small perturbation is added to f (see [13], p. 64); it also equals the multiplicity of the origin as a solution to $\frac{\partial f}{\partial z_0}(z) = \cdots = \frac{\partial f}{\partial z_m}(z)$.

2. The *differential index* is the Poincaré–Hopf index[12] of the vector field ∂f.

3. The *topological index* is the number of spheres μ such that $\vee^\mu S^n$ is the homotopy type of the Milnor fiber $F_{f,0}$; equivalently (as remarked), it is defined as the rank of $\tilde{H}_n(F_{f,0}; \mathbf{Z})$.

[12]The index of a zero z_0 f a vector field v is the degree of the map $z \mapsto v(z)/|v(z)|$ from a small sphere about z_0 to the unit sphere.

4. The *algebraic index* is the dimension of the Milnor algebra

$$\mathcal{A}_f = \mathcal{O}_{f,0}/J_{\partial f}, \tag{40}$$

where $\mathcal{O}_{f,0}$ is the set of germs[13] of polynomial functions $\mathbf{C}^m \to \mathbf{C}$, and $J_{\partial f}$ is the Jacobi ideal $\left\langle \frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_m} \right\rangle$.

Remark. The equality of the algebraic and topological indices was proved by Brieskorn; see [2], p. 154 or [3], p. 573, the latter of which comments specifically on the case $m = 2$. As many as ten different (but equivalent) interpretations of the Milnor number exist, but we shall not enter into their discussion here. The interested reader should consult Martinez [13].

9 Characteristic polynomials of the Milnor fiber

The *characteristic homeomorphism* on the fiber F_0 is given as follows. There is a natural one-parameter family of homeomorphisms

$$h_t \colon F_0 \to F_t \tag{41}$$

for $0 \leq t \leq 2\pi$, where h_0 is the identity and h_t is given by the Covering Homotopy Theorem. Intuitively, $E := S_\epsilon^{2n+1} - K$ has the structure $F_0 \times S^1$ locally, and h_ϵ for ϵ small is given by "sliding" F_0 along this product structure). $h_{2\pi}$ is this characteristic homeomorphism h.

Remark. Brieskorn and Knörrer ([3], p. 574) call this characteristic homeomorphism $h; F_\theta \to F_\theta$ the *geometric monodromy* of the singularity, and they remark that the entire fibration may be essentially reconstructed from the geometric monodromy and a single fiber.

The induced pushforward map on the homology groups is denoted $h_* \colon H_* F_0 \to H_* F_0$.

Definition. This map on the complex homology groups $H_n(F_{f,\theta}; \mathbf{C}) \to H_n(F_{f,\theta}; \mathbf{C})$ is the *Picard–Lefschetz monodromy*.

A result of H. C. Wang (see [14], p. 67) shows that there is an associated exact sequence

$$\cdots \longrightarrow H_{j+1}E \longrightarrow H_j F_0 \xrightarrow{h_* - I_*} H_j F_0 \longrightarrow H_j E \longrightarrow \cdots$$

where I is the identity map; this sequence may be obtained by modification of the ordinary exact sequence of a pair (E, F_0) of fibered space and fiber.

[13] Two functions f, g belong to the same germ if they agree on some open neighborhood of the origin.

Definition. The characteristic polynomial $\Delta(t) = \det(tI_* - h_*)$ of h_* is called the *characteristic polynomial of the Milnor fibration*.

Remark. The polynomial $\Delta(t)$ has degree μ, the rank of the underlying space $H_n F_0$, and has integer coefficients. In the case $n = 1$, Δ is the Alexander polynomial of the corresponding knot K; in the case $n \geq 2$, the Alexander polynomial is a linear factor times Δ (see [14], p. 82). It may be shown (see [15], p. 391) that K has the homology of a sphere S^{2n-1}, and is in fact diffeomorphic to a sphere in the case $n \neq 2$, if and only if $\Delta(1) = \pm 1$.

10 Weighted homogeneous polynomials

A *Brieskorn–Pham polynomial* is any polynomial of the form

$$f(z_0, \ldots, z_n) = z_0^{p_0} + \cdots + z_n^{p_n}, \tag{42}$$

where the exponents p_i are positive integers. Such polynomials, in the case where all the $p_i \geq 2$ (as is required to have an isolated singularity at the origin), are a special case of *weighted homogeneous* polynomials. Conventions differ, but we shall define $f(z)$ to be weighted homogeneous with rational weights $\omega_0, \ldots, \omega_n \in \mathbf{Q} \cap (0, \frac{1}{2}]$ if it is a sum of monomials $z_0^{a_0} \ldots z_n^{a_n}$ where $\sum_{i=0}^n a_i \omega_i = 1$. Such polynomials obey a scaling law

$$\lambda f(z_0, \ldots, z_n) = f(\lambda^{\omega_0} z_0, \ldots, \lambda^{\omega_n} z_n). \tag{43}$$

Proposition 26. *Weighted homogeneous polynomials satisfy the weighted Euler equation*

$$f = \sum_{i=0}^{n} \omega_i z_i \frac{\partial f}{\partial z_i}. \tag{44}$$

Proof. See [13] 2.20, p. 117. □

Remark. It is also possible to define the weights as the reciprocals of these quantities, as is done for example in [15], but this is uncommon in contemporary research. For the origin to be an isolated critical point of such a polynomial, we must have all $0 < \omega_i \leq \frac{1}{2}$.

A concerted study of the Milnor fiber associated with weighted homogeneous polynomials was undertaken by John Milnor and Peter Orlik, generalizing previous work by Brieskorn and Pham. We exposit their work here.

We first define the Milnor number μ of a function f with a singularity at the origin as the local degree of the mapping $z \mapsto \operatorname{grad} f(z)$ near the origin of $\mathbf{C}^m$. We have a preparatory result:

Lemma 27 (Milnor and Orlik)**.** *Let $G\colon \mathbf{C}^m \to \mathbf{C}^m$ be a polynomial mapping with the i-th component ($0 \leq i \leq n = m - 1$) homogeneous of degree d_i, and such that $G^{-1}(0) = 0$. Then* *the local degree of G at 0 is $d_0 \cdots d_n$.*

Proof. See [15] Lemma 1, p. 386. For any i, we may extend the hypersurface $G_i^{-1}(0) := G^{-1}(\{(z_0, \dots, z_n) \in \mathbf{C}^m | z_i = 0\})$ to a hypersurface on the complex projective plane $H_i \subset P^m(\mathbf{C})$. The total intersection multiplicity of $H_0, \dots, H_n$ is the product $d_0 \cdots d_n$ of the degrees corresponding to each H_i. But the intersection $\bigcap_{i=0}^n H_i$ contains only the origin; this one intersection point must thus have degree $d_0 \cdots d_n$. □

This gives the following result on the Milnor number:

Proposition 28 (Milnor and Orlik). *If $f(z_0, \dots, z_n)$ is a weighted homogeneous polynomial with weights $\omega_0, \dots, \omega_n$, then the Milnor number of f is*

$$\mu(f) = \prod_{i=0}^{n} \left(\frac{1}{\omega_i} - 1 \right). \tag{45}$$

Proof. See [15] Theorem 1, p. 2. Write $\omega_i = v_i/u_i$ where $\gcd(u_i, v_i) = 1$, and let $d = \operatorname{lcm}(u_1, \dots, u_m)$. Let $q_i = d\omega_i \in \mathbf{Z}$. Now define $G \colon \mathbf{C}^m \to \mathbf{C}^m$ as

$$G(z_0, \dots, z_n) = (z_0^{q_0}, \dots, z_n^{q_n}). \tag{46}$$

Thus $f \circ G$ is homogeneous of degree g, and the i-th component of $\operatorname{grad} f \circ G$ is homogeneous of degree $d - q_1$. Therefore, by our previous lemma, $\operatorname{grad} f \circ G$ has local degree $(d - q_0) \cdots (d - q_n)$. Local degrees of function compositions are multiplicative, so the local degree of f' is

$$\frac{(d - q_0) \cdots (d - q_n)}{q_0 \cdots q_n} = \left(\frac{1}{\omega_0} - 1 \right) \cdots \left(\frac{1}{\omega_n} - 1 \right), \tag{47}$$

as required. □

Remark. This result is even more remarkable than it may seem; as it may not be obvious *a priori* the weights ω_i of a weighted homogeneous polynomial must be restricted such that $\prod_{i=0}^n (\frac{1}{\omega_i} - 1)$ is an integer.

A couple other results are worth mentioning.

Lemma 29 (Milnor). *If f is weighted homogeneous with weights $\omega_0, \dots, \omega_n$, then the fiber $F = F_0$ of the Milnor fibration is diffeomorphic to $F' = f^{-1}(1) \in \mathbf{C}^m$ (without an intersection with S_ϵ^{2n+1}), and the monodromy h gives a characteristic corresponding map on F. In this case, we may also define the monodromy as*

$$h(z_0, \dots, z_n) = (e^{2\pi i \omega_0} z_0, \dots, e^{2\pi i \omega_n} z_n). \tag{48}$$

Proof. See [14] 9.4, p. 76. □

Proposition 30 (Brieskorn, Pham). *If $f(z_0, \ldots, z_n) = z_0^{a_0} + \cdots + z_n^{a_n}$, then the corresponding characteristic polynomial of the Milnor fibration is*

$$\Delta(t) = \prod (t - \zeta_0 \cdots \zeta_n), \tag{49}$$

where each quantity ζ_i ranges over all a_i-th roots of unity other than 1.

Proof. See [13], p. 134. □

Remark. Martinez [13] computes explicitly the characteristic polynomial for an arbitrary torus link.

11 The classical case: functions of two variables

In the classical case $n = 1$, f is a function of two complex variables, and the Milnor fibers are two-dimensional manifolds whose common boundary K is one-dimensional and therefore possibly a knot. This case has been treated at length by many authors, including Brieskorn and Knörrer [3], and admits a complete classification of singularities and the resulting knots up to topological equivalence, defined here:

Definition. Two functions $f, g\colon \mathbf{C}^2 \to \mathbf{C}$ have *topologically equivalent singularities* at the origin if there are neighborhoods $U_f, U_g \ni 0$ such that a homeomorphism from U_f to U_g restricts to a homeomorphism from $U_f \cap V(f)$ to $U_g \cap V(g)$.

The setting throughout is a polynomial of two complex variables $f(x, y)$ with a singularity at the origin; let $V \subset \mathbf{C}^2$ be its vanishing set. According to Lemma 2, there is some $\epsilon > 0$ such that $(B_\epsilon, V \cap B_\epsilon)$ is homeomorphic to $C(S^3_\epsilon), C(V \cap S^3_\epsilon)$, where C is the cone over X.

Every polynomial f receives an associated set $K = V(f) \cap S^3_\epsilon$. This is a closed manifold of one-real dimension; such manifolds are called *links*; in the case where K has only one connected component and is thus homeomorphic to S^1, it is a *knot*.

Instead of working with intersections with the sphere S^3_ϵ, it is more convenient to work with polydisks. A closed polydisk with multiradius (δ, η) is defined as follows:

$$\overline{D_{\delta,\eta}} = \left\{(x, y) \in \mathbf{C}^2 \middle| |x| \le \delta, |y| \le \eta\right\}. \tag{50}$$

The open polydisk $D_{\delta,\eta}$ is this expression with the $\le$ signs strengthened to $<$; the boundary $\partial D_{\delta,\eta}$ is the union of two solid tori $S^1 \times D^2_{\delta,\eta}$, with one torus corresponding to the set $\{(x, y) | |x| = \delta, |y| \le \eta\}$ and another corresponding to $\{(x, y) | |x| \le \delta, |y| = \eta\}$ (see [8], p. 38–39). One may construct a homeomorphism from $\partial D_{\delta,\eta}$ to S^3 via a similar decomposition of S^3 into two tori: if $S^3_\epsilon = \{(x, y) \in \mathbf{C}^2 | |x|^2 + |y|^2 = \epsilon^2\}$, and $\delta^2 + \eta^2 = \epsilon^2$, then each of

$$T_+ = \left\{(x,y) \in S^3_\epsilon \middle| |y| \leq \eta\right\} \tag{51}$$
$$T_- = \left\{(x,y) \in S^3_\epsilon \middle| |x| \leq \delta\right\} \tag{52}$$

is homeomorphic to a torus, and their union is S^3_ϵ. The spaces S^3 and $D_{\delta,\eta}$, most importantly, may be shown to be interchangeable in the following precise sense:

Proposition 31. *For $\epsilon, \delta, \eta > 0$ sufficiently small and an algebraic set V, some homeomorphism $S_\epsilon \cong \partial D_{\delta,\eta}$ restricts to a homeomorphism $S_\epsilon \cap V \cong \partial D_{\delta,\eta} \cap V$.*

Proof. See [3], p. 419. □

The equation $f(x,y) = 0$ may be solved on the surface of a polydisk with the comparatively simple equations $f(\delta e^{i\theta}, y) = 0$ or $f(x, \eta e^{i\theta}) = 0$; spheres are more difficult.

Denote the surface of a polydisk by Σ. We would now like to calculate the topology of certain algebraic sets corresponding to polynomials of two variables; we may do this by taking their intersections with Σ by a certain process of approximation.

Definition. A polynomial $f(x,y)$ of two complex variables is *weighted homogeneous* if there exist fixed positive rational numbers m, c such that every term in the polynomial has the form $\alpha_{a,b}x^a y^b$ where $a + mb = c$.

Remark. This definition is equivalent to, but more notationally convenient than, the more general definition given previously.

If $\tilde{f}$ is weighted homogeneous with parameters m, c, then $f(x,y) = 0$ can be solved with the form $y = tx^m$ (see [8] 4.3):

$$f(x, tx^m) = \sum \alpha_{a,b} x^a t^b x^{bm} \tag{53}$$
$$= \sum \alpha_{a,b} t^b x^c \tag{54}$$
$$= g(t)x^c, \tag{55}$$

where $g(t) = \sum_{a,b} \alpha_{a,b} t^b$ is a polynomial; if $g(t_0) = 0$, then $y = t_0 x^m$ solves $f(x,y) = 0$. The idea behind Puiseux expansions is to separate a general polynomial f as $f = \tilde{f} + h$, where $\tilde{f}$ is weighted homogeneous and h contains the higher-order terms, in the sense that they have the form $\alpha_{ab}x^a y^b$ where $a+mb$ is greater than the value of this expression on the terms in $\tilde{f}$. (Close to the origin, higher-order terms are of course smaller.) We may then use $\tilde{f}(x,y) = 0$ as an approximate solution to $f(x) = 0$. Joubert ([8], p. 42) describes a general procedure, which rests on the fact that any polynomial can be given a nonzero term of the form $\alpha_{0,b}y^b$ and a weighted homogeneous term with the same weighted degree as $\alpha_{0,b}y^b$ via change of coordinates, and provides examples. It may be shown that via a process of successive approximations, y may be written as a power series $y(x) = \sum_i \alpha_i x^{i/n}$ in fractional powers of x, and that the denominators n are bounded.

Remark. Branches of the functions $x^{1/n}$ must be chosen consistently in Puiseux expansion; for example, an expansion of the form $y = ax^{1/2} + bx^{3/4}$ has four branches, not eight. (That is, we really have $y = ax_4^2 + bx_4^3$, where x_4 is one of the four branches of the function $x^{1/4}$.)

Intuitively, therefore, adding a term to a Puiseux expansion whose exponent has a *larger* denominator than those of the preceding terms will split each the strands of zero solutions on the surface of the polydisk K, whereas adding a term with the same denominator will only perturb K without altering its topology, and such perturbations can be made arbitrarily small and smoothed out by taking the sphere S^3_ϵ to have sufficiently small radius. (See [8], pp. 43–45 for a more detailed and illustrated intuitive explanation.) This notion may be made rigorous.

Remark. It is important to note here that the terms in Puiseux expansions have both increasing exponents and increasing *denominators* of exponents.

Definition. Let $y = \sum \alpha_k x^k, k \in \mathbf{Q}, k \geq 1$ be a Puiseux expansion. The *Puiseux pairs* of f are defined as follows:

1. If all the exponents k are integers, no pairs are defined.
2. If the smallest exponent k_1 is not an integer, write $k_1 = \frac{n_1}{m_1}$ where $\gcd(m_1, n_1) = 1$, and call (m_1, n_1) the first Puiseux pair.
3. If $j \geq 1$ Puiseux pairs are already defined, then let k_{j+1} be the smallest exponent that cannot be written in the form $q/m_1 \cdots m_j$. Then, write

$$k_{j+1} = \frac{n_{j+1}}{m_1 \cdots m_{j+1}} \tag{56}$$

   with $\gcd(m_{j+1}, n_{j+1}) = 1$, and define (m_{j+1}, n_{j+1}) to be the next Puiseux pair.

Proposition 32 (Pham)**.** *The topology of a knot depends only on its Puiseux pairs.*

Proof. See [8] 4.4.2 or [3], p. 411. □

This result, together with another result that two knots with identical Alexander polyomials must be homeomorphic, leads to a complete classification of the knots of weighted homogeneous polynomials.

Proposition 33 (Lê)**.** *The Alexander polynomial of a knot K with Puiseux pairs $(m_1, n_1), \ldots,$ (m_s, n_s) is*

$$\Delta_K(t) = \prod_{i=1}^{s} P_{\lambda_i, n_i}(t^{\nu_{i+1}}), \tag{57}$$

where $\nu_i = n_i \cdots n_s$ *for* $1 \leq i \leq s$, $\nu_{s+1} = 1$, $\lambda_1 = m_1$, $\lambda_i = m_i - m_{i-1}n_i - \lambda_{i-1}n_1 n_{i-1}$ *for* $2 \leq i \leq s$, *and the terms in the product are given by*

$$P_{\lambda,n}(t) = \frac{(t^{\lambda n} - 1)(t-1)}{(t^{\lambda} - 1)(t^n - 1)}. \tag{58}$$

Proof. See [10] 2.4.10, p. 293 (which gives this result as a recurrence relation on s) or [13] 4.80, p. 323. □

Remark. It is possible to show, by computing the Alexander polynomials using this formula of Lê and showing them to be equal, that the two-variable Brieskorn–Pham polynomials $f(z_1, z_2) = z_1^p + z_2^q$ give the *torus link* $T_{p,q}$. For $p = 2, q = 3$, the case investigated by Brauner, this is indeed the trefoil knot; its Alexander polynomial is $\Delta_K(t) = t^2 - t + 1$ (see [13], p. 713).

12 Quantum field theory on the torus

A key notion in quantum field theory is that of gauge invariance: physical variables underlying certain systems can be altered in certain continuous ways without changing the resulting dynamics.[14] In quantum field theory, system dynamics are determined by a Lagrangian, each of whose terms corresponds to a boson (which takes integer values of spin and can occupy the same state) or to fermions (which takes half-integer values of spin and cannot occupy the same state). A theory is supersymmetric if a continuous gauge transformation to the Lagrangian is possible that interchanges the terms for bosons and fermions. In such a system, every particle has a corresponding supersymmetric partner particle.

Equivalently by Noether's theorem, which associates every gauge symmetry with a corresponding conserved physical quantity, we can define supersymmetry not as a gauge property but as the existence of an operator Q, called the *supercharge*, whose effect is to exchange bosonic and fermionic states.

Among the first applications of knot theory to quantum field theory was Witten's [20] paper on the relationship between the Jones polynomial and certain problems in Yang–Mills theory. Witten began by noting a certain analogy between the subjects: certain important invariants in knot theory, such as the Jones polynomial, must be computed from counting crossings on two-dimensional projections of the knot and then proving laboriously that the invariants therefrom constructed are in fact independent. Witten compares this to early work in quantum field theory, which tried to prove relativistic invariance of various aspects of its theory without the modern, manifestly relativistically invariant building blocks of quantum field theory.

[14] A prototypical example comes from classical mechanics: as only changes in energy, not total energies themselves, are physically meaningful, arbitrary scalar constants may be added to the potential energy of a system without altering the dynamics.

Witten's paper dealt with the problem of quantization of the system with a Lagrangian resulting from the integral of the Chern–Simons form

$$\mathcal{L} = \frac{k}{4\pi} \int_M \mathrm{Tr}(A \wedge dA + \frac{2}{3} \wedge^3 A). \tag{59}$$

Instead, the work in quantum field theory with which we concern ourselves here was begun by Arthur Jaffe and continued by his student, Robert Martinez. Consider, as exposited in [7], a system defined on a one-dimensional circular physical space, with Hamiltonian H, a scalar bosonic field $\varphi(x)$ with components $\varphi_i(x)$ for $1 \leq i \leq n$, and a vector Dirac field $\psi(x)$ with components $\psi_{\alpha,i}(x)$, where $\alpha = 1, 2$ and $1 \leq i \leq n$. These states evolve in time as

$$\varphi(x, t) = e^{itH} \varphi(x) e^{-itH} \tag{60}$$

and analogously for ψ. Define a bosonic conjugate field $\pi(x) = [iH, \phi(x)^*]$ where $[A, B] = AB - BA$ is the commutation bracket.

The underlying one-particle space of states is

$$\mathcal{K} = \bigoplus_{i=1}^{n} L^2(S^1, dx) \oplus L^2(S^1, dx) \tag{61}$$

of complex distributions in position space; the two L^2 distributions are for the two components of each ψ_i. The Fock space of particle states shall be denoted $\mathscr{H} = \mathscr{H}^b \otimes \mathscr{H}^f$, where $\mathscr{H}^b = \bigoplus_{n \geq 0} \mathcal{K}^{n \otimes s}$ is the bosonic Fock space for symmetric states of one or more particles, and $\mathscr{H}^f = \bigoplus_{n \geq 0} \mathcal{K}^{n \wedge}$ is the Fock space for antisymmetric states of one or more particles.

As in ordinary quantum field theory, the momentum operator P commutes with H and is the generator of translations:

$$e^{-i\sigma P} \varphi(x) e^{i\sigma P} = \varphi(x + \sigma) \tag{62}$$

and analogously for π and ψ.

The group of twists of the field (corresponding to phase shifts if the field is rotated by a certain angle) has a generator J; twisting the i-th component of a field induces a change given by a constant *twisting angle* Ω. Specifically, for the bosonic field,

$$e^{i\theta J} \varphi_i(x) e^{-i\theta J} = e^{i\theta \Omega_i^b} \varphi_i(x) \tag{63}$$

for a twist angle Ω_i^b. For the fermionic field,

$$e^{i\theta J} \psi_{\alpha,i}(x) e^{-i\theta J} = e^{i\theta \Omega_{\alpha,i}^f} \psi_{\alpha,i}(x) \tag{64}$$

for a constant twist angle $\Omega^f_{\alpha,i}$. The twist angles characterize the generator J up to an additive constant, which we can choose for convenience without altering the system's dynamics. For convenience, let this constant be $\hat{c}/2$, where

$$\hat{c} = \sum_{i=1}^{n} \left(\Omega^f_{2,i} - \Omega^f_{1,i} \right). \tag{65}$$

Now let the circumference of the circle S^1 be ℓ. We define a twist quantum field on ℓ as a quantum field defined above plus twisting angles $\chi^b_i, \chi^b_{\alpha,i}$ such that the following relations hold for a full twist of the circle.

$$\varphi_i(x+\ell) = e^{i\chi^b_i}\varphi_i(x) \tag{66}$$

$$\pi_i(x+\ell) = e^{-i\chi^b_i}\pi_i(x) \tag{67}$$

$$\psi_{\alpha,i}(x+\ell) = e^{i\chi^f_{\alpha,i}}\psi_{\alpha_i}(x). \tag{68}$$

We further assume that none of the twisting angles $e^{i\chi}$ equals 1; that is, none of the χ^b_i or $\chi^f_{\alpha,i}$ is a multiple of 2π.

Suppose now that there exists a polynomial V of n complex variables and degree $\tilde{n} \geq 2$, called the *superpotential*, that governs the nonlinear dynamics of this system. Suppose further that V is homogeneous with weights Ω_i, $1 \leq i \leq n$. V is called the superpotential because it defines the Hamiltonian, which has a free term H_0 and an interaction term governing the coupling of the bosonic field to the Dirac field as follows:

$$\begin{aligned} H_I(x) = &\sum_{j=1}^{n} |V_j(\varphi(x))|^2 \\ &+ \sum_{i,j=1}^{n} \left(\psi_{i,1}(x)\psi_{j,2}(x)^* V_{ij}(\varphi(x)) + \psi_{i,2}(x)\psi_{j,1}(x)^* V_{ij}(\varphi(x))^* \right). \end{aligned} \tag{69}$$

We have defined a *Generalized Wess–Zumino Model*. The first term in the Hamiltonian is invariant both under translations and twists as long as the twisting angles χ^b_i and twist parameters Ω^b are proportional to the weights Ω_i. In fact, set $\chi^b_i = \Omega^b_i\phi = \Omega_i\phi$, where $0 < \phi \leq \pi$.

We also want the second term in the Hamiltonian to be invariant under twists and translations, which means that H_I commutes with both J and P. It turns out that for this to hold, $\Omega^f_{1,i} - \Omega_{2,j} + 1 - \Omega_i - \Omega_j$ and $\chi^f_{1,i} - \chi^f_{2,j} + \phi - \Omega_i\phi - \Omega_j\phi$ must both be multiples of 2π—we may take this to mean "must equal zero," because changes of twist parameters by multiples of 2π are physically irrelevant. This is solved as follows:

$$\Omega^f_{1,i} = \Omega_i - \tfrac{1}{2} + \epsilon \tag{70}$$

$$\Omega^f_{2,i} = -\Omega_i + \tfrac{1}{2} + \epsilon \tag{71}$$

$$\chi^f_{\alpha,i} = \Omega^f_{\alpha,i}\phi + \nu. \tag{72}$$

The real parameters ϵ and ν are the same for each twist angle. The additive constant $\hat{c}$ defined previously for J, in terms of the weights, is simply

$$\hat{c} = \sum_{i=1}^{n}(1 - 2\Omega_i). \tag{73}$$

Definition. The sum $\hat{c}$ is the *central charge*.

If we define a grading $\Gamma = (-I)^{N_f}$ on the Fock space $\mathscr{H} = \mathscr{H}^b \oplus \mathscr{H}^f$, where N_f is the fermionic number operator, then Γ commutes with the Hamiltonian $H(V)$. If we assign $\epsilon = \pm\frac{1}{2}$ and $\mu = 0$ in the twist parameters, then we can define supercharge operators $Q_\pm$ such that $Q_\pm\Gamma + \Gamma Q_\pm = 0$. The choices $\epsilon = \frac{1}{2}$ and $\epsilon = -\frac{1}{2}$ correspond to Q_+ and Q_- respectively, which satisfy

$$Q_+^2 = H + P \tag{74}$$

$$Q_-^2 = H - P. \tag{75}$$

The introduction of a twist, therefore, *breaks* one of the two supersymmetries. In this way, the supersymmetric quantum field theory in question is *partially broken*.

To proceed, we must introduce a condition called the *elliptic bound* on the superpotential V. Given any multiderivative $\partial^\alpha = \partial^{i_1}\cdots\partial^{i_m}$, and letting $|\partial V|^2 = |\operatorname{grad} V|^2$, the elliptic bound is the existence of some finite M for any $\epsilon > 0$ such that the following two conditions hold:

$$|\partial^\alpha V| \leq \epsilon|\partial V|^2 + M \tag{76}$$

$$|z|^2 + V \leq M\left(|\partial V|^2 + 1\right). \tag{77}$$

This elliptic bound ensures that the Hamiltonian has a discrete spectrum with rapidly increasing eigenvalues. If V is weighted homogeneous (with the weight ω_i identified with Ω_i) and satisfies the elliptic bound (e.g., when V has an isolated critical point at the origin), then one says that V satisfies the *standard hypotheses*.

We have the following existence theorem for the supercharge.

Proposition 34 (Jaffe). *If the potential V satisfies the standard hypotheses, then there exists a self-adjoint operator Q that commutes with the group $e^{-i\sigma P - i\theta J}$ of twists and translations, anticommutes with the grading Γ, and such that $H = Q^2 - P$. The spectrum of the Hamiltonian*

$H(V)$ is bounded from below, and the heat-kernel operator $e^{-\beta H}$ is a trace-class operator[15] *for $\beta > 0$ (corresponding to positive temperature).*

Proof. See [7] 1.3, p. 1420. □

13 The elliptic genus of the Wess–Zumino model

Definition. In topology, the *multiplicative genus* is a function ϕ that assigns to every closed oriented smooth manifold M^n of dimension n an element of a commutative $\mathbf{Q}$-algebra Λ with unit, and that satisfies the following axioms (see [16]):

1. If M^n and N^n are disjoint, then $\phi(M^n \cup N^n) = \phi(M^n) + \phi(N^n)$.
2. $\phi(M^n \times V^m) = \phi(M^n)\phi(V^m)$.
3. If M^n is the oriented boundary of a manifold δW^{n+1}, then $\phi(m) = 0$.

Remark. The first and third axioms imply that ϕ is invariant under cobordism.

Definition. The map ϕ is an *elliptic genus* if it is zero on manifolds of the form $\mathbf{C}P(\xi)$, where ξ is an even-dimensional complex vector bundle over a closed oriented manifold, and $\mathbf{C}P(\xi)$ denotes the associated projective bundle.

Jaffe considers the following partition function:

$$\mathfrak{Z}^V = \mathrm{Tr}_{\mathscr{H}} \left(\Gamma \exp \left(-i\theta J - i\sigma P - \beta H \right) \right). \tag{78}$$

The work of Martinez [13] indicates precisely how this object is an elliptic genus in the mathematical sense defined above. We show, for example, that the genus is a cobordism equivalent, and that the multiplication axiom follows from its behavior under Sebastiani–Thom summation (see Proposition 41).

Definition. The Gaussian Borel measure $\gamma^n \colon B_0(\mathbf{R}^n) \to [0, +\infty)$, where B_0 denotes the space of Borel sets, on a real space $\mathbf{R}^n$ is defined as

$$\gamma^n(A) = \frac{1}{(2\pi)^{n/2}} \int_A \exp\left(-\frac{1}{2} |x|^2 \right) d\lambda. \tag{79}$$

where λ is the usual Lebesgue measure.

Remark. This definition has been generalized to operator-valued distributions in the Constructive Quantum Field Theory literature, e.g. [5].

[15] An operator A on a separable Hilbert space with orthonormal basis $e_1, e_2, \ldots$ is a trace-class operator if $\sum_k \langle |A| e_k, e_k \rangle = \sum_k \left\langle (A^*A)^{1/2}, e_k \right\rangle$ is finite. One may define a basis-independent trace $\mathrm{Tr}\, A = \sum_k \langle A e_k, e_k \rangle$ for such operators.

Proposition 35 (Jaffe). *Under the standard hypotheses, there is a positive and countably additive Borel measure, other than the Gaussian measure, on the space $\mathcal{S}' = \mathcal{S}'(\mathbf{T}^2)$ of distributions on the 2-torus $T^2 = S^1 \times S^1$ and an integrable, renormalized Fredholm determinant*[16] $\det_3$ *such that*

$$\mathfrak{Z}^V = \int_{\mathcal{S}'} \det_3 d\mu. \tag{80}$$

Proof. See [7] 2.2, p. 1420. □

This measure μ is in fact constructed from the Gaussian measure as $d\mu = e^{-S} d\gamma$, where γ is the Gaussian measure and S is a real-valued functional corresponding to the physical action.

Now, define the additional parameter $\tau = \frac{\sigma + i\beta}{\ell} \in \mathbf{H}$ in terms of the translation σ and Boltzmann weight β. ($\mathbf{H}$, of course, is the complex upper half-plane.)

Proposition 36 (Jaffe). *For a fixed twist parameter ϕ and twist θ, the elliptic genus is a holomorphic function of τ alone.*

Proof. see [7] 2.4, p. 1420. □

In fact, the elliptic genus only depends on V via its weights Ω_i, and may be calculated as follows:

Proposition 37 (Jaffe). *Under the standard hypotheses, $\mathfrak{Z}^V$ may be written*

$$\mathfrak{Z}^V(\tau, \theta, \phi) = e^{i\theta\hat{c}/2} \prod_{i=1}^{n} \prod_{k \geq 0} \frac{(1 - y^{-(1-\omega_i)q^k})(1 - y^{1-\omega_i} q^{k+1})}{(1 - y^{-\omega_i} q^k)(1 - y^{\omega_i} q^{k+1})} \tag{81}$$

$$= y^{\hat{c}/2} \prod_{i=1}^{n} \frac{\vartheta_1(\tau, (1 - \Omega_i)(\theta - \phi\tau))}{\vartheta_1(\tau, \Omega_i(\theta - \phi\tau))}, \tag{82}$$

where ϑ_1 is a Jacobi theta function of the first kind, and new variables are defined as follows: $q = e^{2\pi i \tau}$ is a nome function (note that as $\tau \in \mathbf{H}$, so $|q| < 1$), $z = \frac{\theta - \tau\phi}{2\pi}$, and $y = e^{2\pi i z}$ is another nome function.[17]

Proof. See [7] 3.1, p. 1422, or [13], p. 589. □

Remark. The elliptic genus obeys an elegant transformation property. If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in$ $\mathrm{SL}(2, \mathbf{Z})$ is a matrix with integer coefficients such that $ad - bc = 1$, and the parameters τ, θ, ϕ are transformed accordingly as $\tau' = \frac{a\tau + b}{c\tau + d}$, $\theta' = \frac{\theta}{c\tau + d}$, and $\phi' = \frac{\phi\tau}{a\tau + b}$, then

$$\mathfrak{Z}^V(\tau', \theta', \phi') = \exp\left[2\pi i \left(\frac{\hat{c}}{8}\right)\left(\frac{c(\theta - \phi\tau)^2}{c\tau + d}\right)\right] \mathfrak{Z}^V(\tau, \theta, \phi). \tag{83}$$

[16] The Fredholm determinant is a similar generalization of the notion of determinants of vector operators to the space of operators on a separable Hilbert space.

[17] These conventions are from [13] and [12], not [7].

Limits as the parameters θ, q, ϕ tend to zero correspond to the untwisted quantum-mechanical theory. These limits do not generally commute. Several of them, however, equal the *Witten index* or *supertrace* of the supercharge operator, defined as $\operatorname{ind} Q_+ = \dim\ker Q_+ - \dim\operatorname{coker} Q_+$.

Corollary 38 (Jaffe)**.** *The following equation holds:*

$$\begin{aligned}
\operatorname{ind} Q_+ &= \lim_{\theta\to 0}\lim_{\phi\to 0} \mathfrak{Z}^V && (84)\\
&= \lim_{\phi\to 0}\lim_{\theta\to 0} \mathfrak{Z}^V && (85)\\
&= \lim_{\theta\to 0}\lim_{q\to 0}\lim_{\phi\to 0} \mathfrak{Z}^V && (86)\\
&= \prod_{i=1}^{n}\left(\frac{1}{\Omega_i} - 1\right). && (87)
\end{aligned}$$

Proof. See [7], p. 1422. □

Once again, note the similarity in form between this result and that of the Milnor number of a weighted homogeneous polynomial; this suggests a deeper relationship.

Remark. In a supersymmetric theory, every nonzero eigenvalue of the Hamiltonian H corresponds to an equal number of bosonic and fermionic eigenstates, with the correspondence given by the action of the supercharge Q. Therefore, the Witten index Q is the number of bosonic minus the number of fermionic *ground states*.

14 Hilbert–Poincaré series of the Milnor algebra

The work of Robert Martinez has proven several deep results at the intersection of the Milnor theory of the superpotential V and the corresponding quantum field theory.

Definition. Let $A = \bigoplus_{i\in\mathbf{Z}} A_i$ be an algebra graded over the integers (i.e. $A_iA_j \subseteq A_{i+j}$). The associated *Hilbert–Poincaré series* is the formal series

$$P_A(t) = \sum_{i\in\mathbf{Z}} (\dim A_i)t^i. \tag{88}$$

Remark. The value (or limit) $P_A(1)$ is the dimension of the underlying algebra. This series, as it should not be hard to believe, is additive over direct sums and multiplicative over tensor products.

The Milnor algebra associated to a function $f\colon \mathbf{C}^m \to \mathbf{C}$ has a natural grading over $\mathbf{Z}$ that associates to every weighted homogeneous polynomial its weighted degree; as with ordinary degrees, weighted degrees of the products of two polynomials add. It therefore also has a Hilbert–Poincaré series, which may be computed (see [13], p. 126) as

$$P_{\mathcal{A}_f}(t) = \prod_{i=1}^{m} \frac{1 - t^{d-q_i}}{1 - t^{q_i}}, \tag{89}$$

where d is the smallest integer such that $d\omega_i \in \mathbf{Z}$ for all i, and $q_i = d\omega_i$. (Recall our assumption $\omega_i \leq 1/2$, which implies that $q_i \leq d/2$.)

Remark. Because the underlying Milnor algebra for a weighted homogeneous polynomial is finite-dimensional (as will be clear momentarily), the Hilbert–Poincaré series terminates, and the above rational expression simplifies to a polynomial.

The above form for the Hilbert–Poincaré formula cannot be evaluated directly at $t = 1$; we can, however, evaluate the limit of each term at $t \to 1$ with l'Hôpital's rule:

$$\lim_{t\to 1} \frac{1 - t^{d-q_i}}{1 - t^{q_i}} = \lim_{t\to 1} \frac{(d - q_i)t^{d-q_i-1}}{q_i t^{q_i-1}} = \frac{d - q_i}{q_i} = \frac{1}{\omega_i} - 1. \tag{90}$$

Because the limit of a product of (finite-valued) functions at a point is the product of the limits, one has

$$\lim_{t\to 1} P_{\mathcal{A}_f}(t) = \prod_{i=1}^{m} \left(\frac{1}{\omega_i} - 1 \right). \tag{91}$$

Note the coincidence of this algebraic result with that result derived by Milnor and Orlik, establishing a link between topology and algebra.

15 Links between topological and quantum invariants

First, a few preliminary definitions are needed.

Definition. A *Hodge structure of weight* n (see [17], p. 1) is an Abelian group $H_{\mathbf{Z}}$ together with a decomposition of the complex vector space $H = H_{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathbf{C}$ into subspaces $H^{p,q}$, with $p+q = n$, such that $H^{p,q} = \overline{H^{q,p}}$ and $\oplus H^{p,q} = H$. Equivalently, the Hodge structure may be given as a filtration of H into complex subspaces $F^p H$ that satisfy the condition that for p, q, such that $p + q = n + 1$, $F^p H \cap \overline{F^q H} = 0$ and $F^p H \oplus \overline{F^q H} = H$.

This concept and the following are used especially often in the context of decomposing homology groups of algebraic varieties.

Definition. Let $R \subset \mathbf{C}$ be a noetherian subring such that $R \times \mathbf{Q}$ is a field, and let V_R be a finitely generated R-module. An *R-mixed Hodge structure* on V_R (see [17], p. 62) is a set of two filtrations: an ascending[18] *weight filtration* $W_\bullet$ on $V_R \otimes_R (R \otimes \mathbf{Q})$ and a descending[19] *Hodge filtration* $F^\bullet$ on $V_{\mathbf{C}} = V \otimes_{\mathbf{R}} C$. Furthermore, the Hodge filtration

[18] i.e. $W_i \subseteq W_{i+1}$
[19] i.e. $F^i \supseteq F^{i+1}$

induces a pure Hodge structure of weight k on each graded piece $\mathrm{Gr}_k^W(V_R \otimes \mathbf{Q}) = W_k/W_{k-1}$ defined by the weight fibration.

Steenbrink [19] calculated the existence of a mixed Hodge structure on the cohomology of the Milnor fiber of a weighted homogeneous singularity. Steenbrink's immediate motivation was to provide a proof of a conjecture by Arnol'd concerning the signs of the eigenvalues of the intersection form on the middle cohomology group of the Milnor fiber of a weighted homogeneous polynomial.

The coincidence between the formulas for a certain limit of the elliptic genus and the Milnor number of the generating superpotential function suggests that one can be derived from the other. This is in fact true; the algebraic Milnor number can be computed from the Hilbert–Poincaré series, which can itself be computed from the so-called *Steenbrink exponents* of the elliptic genus, which are topological invariants. The following exposition follows the work of Martinez [13].

Definition. For a weighted homogeneous polynomial $f\colon \mathbf{C}^m \to \mathbf{C}$, define the *Steenbrink series* of f as

$$\mathrm{Sp}(f;t) = t^{\sum_{i=1}^m \omega_i} \overline{P}_{\mathcal{A}_f}(t) \tag{92}$$

$$= \prod_{i=1}^m \frac{t^{\omega_i} - t}{1 - t^{\omega_i}} \tag{93}$$

$$= \sum_{j=1}^{\mu} t^{\gamma_j} \tag{94}$$

where $\overline{P}_{\mathcal{A}_f}(t)$ is the reduced Poincaré–Hilbert series, defined in [13], p. 136 as

$$\overline{P}_{\mathcal{A}_f}(t) = P_{\mathcal{A}_f}(t^{1/d}) = \prod_{i=0}^n \frac{1 - t^{1-\omega_i}}{1 - t^{\omega_i}},$$

where d is defined as before as the least integer such that $\omega_i d \in \mathbf{N}$ for all i. The quantities γ_j are the *Steenbrink exponents*.

Remark. See [13], p. 221. Note that for known weights, the Hilbert–Poincaré and the Steenbrink series determine each other.

In a groundbreaking work, Martinez [13] proves the following classification of the Wess–Zumino models by way of considering the associated characteristic polynomial and Steenbrink series of the superpotentials.

Proposition 39 (Martinez). *Let $f\colon \mathbf{C}^m \to \mathbf{C}$ be a non-degenerate weighted homogeneous polynomial satisfying the standard hypotheses of* [7]. *Then the elliptic genus $\mathfrak{Z}^f$ determines the reduced Alexander polynomial of the algebraic link K_f. Moreover, if $m = 2$ and $K_f \subset S^3$ is a knot, then the Alexander polynomial is a cobordism*[20] *and isotopy invariant.*

[20]Two manifolds of dimension n are *cobordant* if their union is the boundary of a compact manifold of dimension $n+1$.

Proof. By Proposition 37, the genus has the following formula (with new parameters defined as before):

$$\mathfrak{Z}^f(z,\tau) = y^{\hat{c}/2} \prod_{i=1}^{m} \prod_{k\geq 0} \frac{(1-y^{-(1-\omega_i)}q^k)(1-y^{1-\omega_i}q^{k+1})}{(1-y^{-\omega_i}q^k)(1-y^{\omega_i}q^{k+1})}, \tag{95}$$

We ultimately care about the limit $q \to 0$, so we can separate the terms dependent on and independent of q and get

$$\mathfrak{Z}^f(z,\tau) = y^{-m/2} \prod_{i=1}^{m} \frac{y^{1-\omega_i}-1}{1-y^{-\omega_i}} + O(q); \tag{96}$$

this product is the Steenbrink series

$$\mathrm{Sp}(f;y) = \prod_{i=1}^{m} \frac{y^{1-\omega_i}-1}{1-y^{-\omega_i}} = \sum_{j=1}^{\mu} y^{\gamma_j}. \tag{97}$$

Steenbrink [19] and Martinez [13] show that the characteristic polynomial Δ_{h_*} of the Lefschetz–Picard monodromy is determined by the spectrum $\mathrm{Sp}(f) = \{\gamma_j\}$ of the Steenbrink sequence as

$$\Delta_{h_*}(t) = \prod_{j=1}^{\mu} (t - e^{2\pi i \gamma_j}). \tag{98}$$

A lemma of Milnor ([14], p. 82) shows that this is related to the Alexander polynomial $\Delta_K(t_1,\ldots,t_n)$ of K by the relation

$$\pm t^i \Delta_{h_*}(t) = (t-1)^{1-\delta_{n,1}} \Delta_K(t,\ldots,t), \tag{99}$$

where δ is the Kronecker delta function. Therefore, Δ_{h_*} completely determines the Alexander polynomial in the case $n = 1, m = 2$. A result of Noguchi shows that the Lefschetz zeta function of the knot can then be written as

$$\zeta_{K_f}(t) = \exp\left(\sum_{k\geq 0} \Lambda(h^{\circ k}) \frac{t^k}{k}\right) \tag{100}$$

$$= \prod_{\ell \geq 0} \det(I - t h_{*,\ell})^{(-1)^{\ell+1}} \tag{101}$$

where $V_{f,1} = f^{-1}(1)$ is diffeomorphic to the Milnor fiber by Lemma 29, h is the geometric monodromy $h(z) = (e^{2\pi i \omega_1} z_1, \ldots, e^{2\pi i \omega_m} z_m)$ chosen in accordance with Lemma 29, $h^k_{*,\ell}$ denotes the induced Lefschetz–Picard monodromy on the ℓ-th homology group

$$h^k_{*,\ell} \colon H_\ell(V_{f,1}; \mathbf{Q}) \to H_\ell(V_{f,1}; \mathbf{Q}), \tag{102}$$

and Λ denotes the Lefschetz number,

$$\Lambda(h^{\circ k}) = \sum_{\ell \geq 0} (-1)^{\ell} \operatorname{tr} h^{k}_{*,\ell}, \tag{103}$$

which, by Lemma 24, equals the Euler characteristic $\chi_k = \left|\left\{z \in V_{f,1} \middle| h^{\circ k}(z) = z\right\}\right|$. This gives the formula

$$\zeta_K(t) = (-1)^{\mu n}(1-t)^{-1}\Delta_{h_*}(t). \tag{104}$$

Finally, if K is a knot in S^3, then the work of Lê ([10]) (Proposition 33) gives a complete classification of such knots up to cobordism and isotopy, depending only on their Alexander polynomials. This completes the proof of the theorem. □

Corollary 40. *The Generalized Wess-Zumino models admit classification according to the spectrum of the corresponding Picard–Lefschetz monodromy and the topology of the corresponding algebraic links.*

Proof. This follows from the previous results and from the complete classification of algebraic links (e.g. in [10]). □

The space of models may be further characterized with a useful property of the Milnor fiber. In this way, the features of the elliptic genus, as defined by Ochanine, become clear.

Definition. If $f\colon X \to \mathbf{C}$ and $g\colon Y \to \mathbf{C}$ are two maps, then the *Sebastiani–Thom summation* $f \boxplus g\colon X \times Y \to \mathbf{C}$ is the map $f \circ \pi_X + g \circ \pi_Y$, where π_X, π_Y are the projection maps of $X \times Y$ onto X and Y respectively.

Proposition 41 (Sebastiani, Thom)**.** *The monodromy of $f \boxplus g$ is the tensor product of those of f and g.*

Proof. This is the main result of [18]. □

The models generated by two superpotentials f and g, with Fock spaces $\mathscr{H}_f$ and $\mathscr{H}_g$, respectively, may be combined to give another model on the tensor product Fock space $\mathscr{H}_f \otimes \mathscr{H}_g$.

Proposition 42 (Martinez)**.** *The elliptic genus of the Sebastiani-Thom superpotential $f \boxplus g$ on a Fock space $\mathscr{H}_f \otimes \mathscr{H}_g$ is given by product of the elliptic genera of f and g on the Fock spaces $\mathscr{H}_f$ and $\mathscr{H}_g$, respectively. In particular, there is a natural monoidal structure on the space of Generalized Wess-Zumino models under the Sebastiani-Thom sum.*

Proof. See Martinez [13], p. 701. □

Martinez also computes a host of invariants of algebraic links in terms of the weights of the corresponding weighted homogeneous singularity, including the delta invariant, branch number, geometric genus, and signature. For space considerations, we discuss only a few of these.

Definition. The *delta invariant* of a function $f\colon \mathbf{C}^2 \to \mathbf{C}$ is the number of double points in the plane curve f. If f is a nondegenerate and square-free weighted-homogeneous polynomial, then this equals the number of positive lattice points in the right triangle with vertices $(0,0)$, $(\frac{1}{\omega_1}, 0)$, and $(0, \frac{1}{\omega_2})$.

Remark. Martinez ([13], p. 358) has a proof of the above statements based on lattice-point enumeration and gives an explicit formula for $\delta(f)$ in terms of the weights of f.

Proposition 43 (Jaffe, Martinez). *The Fredholm index* $\operatorname{ind} Q_+$ *of the (unbroken) supercharge* Q_+ *resulting from superpotential* f *(viewed a complex polynomial) is equal to the Milnor number* $\mu(f)$*, namely,*

$$\operatorname{ind} Q_+ = \lim_{\theta,\phi \to 0} \mathfrak{Z}^f(z,\tau) = \prod_{i=1}^{m} \left(\frac{1}{\omega_i} - 1 \right) = \mu(f), \tag{105}$$

which is equal to $2\delta(f) - r(f) + 1$*, where* $r(f)$ *is the branch number of* f *at the origin and* $\delta(f)$ *is the delta invariant of* f*.*

Proof. See [13], p. 697. □

Remark. The equation $\mu(f) = 2\delta(f) - r(f) + 1$ is known as the *Milnor-Jung formula*. We can say more for torus links, the knots that result from singularities of the form $f(x,y) = x^p + y^q$, where $p, q \geq 2$. The Thom Conjecture, proved by Kronheimer and Mrowka ([13], p.372), implies that the integer $\delta(f)$ is the *unknotting number* of the corresponding link K of the singularity f. Combining this result with that of Martinez, it follows that the elliptic genus $\mathfrak{Z}^f$ yields *knot invariants*, opening up new avenues to study algebraic links using quantum methods. More research in this exciting new field is therefore merited.

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