Loop Spaces

Fall 2020 Tutorial

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> My early 1970's work was then viewed as "too categorical" by older algebraic topologists. In retrospect, it was not nearly categorical enough for intuitive conceptual understanding. — J. P. May

If X is a space equipped with a base point, then its loop space ΩX is the space of all loops in X starting and ending at the base point. This definition implicitly shows up in the definition of the fundamental group, as $\pi_1 X$ is simply the connected components of ΩX.

Just as $\pi_1 X$ is a group, the composition of loops gives ΩX a sort of multiplication. It is what we call an $\mathbb{A}_\infty$-group. It turns out *every* $\mathbb{A}_\infty$ group is of the form ΩX. Moreover, the definition of an $\mathbb{A}_\infty$ is fairly concrete. For example, every topological group G is an $\mathbb{A}_\infty$-group. This gives us the existence of a space BG such that $\Omega BG \simeq G$.

There is a similar statement relating iterated loop spaces $\Omega^n X$ and "infinite loop spaces" to $\mathbb{E}_n$- and $\mathbb{E}_\infty$-groups.

The point of the theorem is not so much that we can find loop spaces this way (though this is a legitimately useful application). Rather, it connects the *topological* notion of being a loop space and the *algebraic* notion of being a group. This gives us a solid way to understand spaces that we know to be loop spaces, such as BU and Eilenberg–Maclane spaces.

Apart from direct applications, the ideas and techniques used in the proof are extremely important and influential in algebraic topology. In the tutorial, we will build from ground up these techniques, assuming only minimal prerequisites. If time permits, we will discuss some applications of the theory. Of course, the content will be adjusted based on the background of the participants.

Specifically, I intend to cover the following topics in the tutorial:

1. Elementary category theory (categories, (co)limits, adjoint functors)
2. Some categorical algebra (monads and operads)
3. Basic homotopy theory (fibrations, cofibrations)
4. The bar construction and simplicial methods

We will also do a fair amount of hands-on down-to-earth calculations in the course of proving the theorem — it is not all just abstract nonsense!

Prerequisites: Familiarity with fundamental groups and singular homology. Know what groups, rings and modules are.