Deformation of Galois Representations

Chung Pang Mok

1. Introduction

As before, let p be an odd prime, N an integer prime to p. Let S be the set of primes dividing N, together with p and ∞. Recall that $G_{\mathbf{Q},S}$ is the Galois group of the maximal extension of $\mathbf{Q}$ unramified outside S. Consider a representation:

$$\bar{\rho} : G_{\mathbf{Q},S} \to \mathrm{GL}_2(\mathbf{F}_p)$$

which we assume to be absolutely irreducible (called a residual representation in what follows). One is interested in parametrising the representations

$$\rho : G_{\mathbf{Q},S} \to \mathrm{GL}_2(\mathbf{Z}_p)$$

such that the reduction of ρ mod p is equivalent to $\bar{\rho}$. Such representations are called *deformations* of $\bar{\rho}$ (in what follows, an over-bar means reduction mod p).

This question is closely connected with that of congruence between modular forms. Indeed, let $f \in S_k(\Gamma_0(Np^r), \mathbf{Z}_p)$ be a cuspidal eigenform (here $k \geq 2, r \geq 1$). Work of Shimura, Deligne, shows that one can associate to f a Galois representation:

$$\rho_f : G_{\mathbf{Q},S} \to \mathrm{GL}_2(\mathbf{Z}_p)$$

such that

$$\mathrm{tr}(\rho_f(\mathrm{Frob}_l)) = a(l, f), \;\; \det(\rho_f(\mathrm{Frob}_l)) = l^{k-1}, \text{ for } l \not| Np.$$

¿From now on take the residual representation to be $\bar{\rho}_f$ (assumed to be absolutely irreducible). Then for any eigenform $g \in S_k(\Gamma_0(Np^r), \mathbf{Z}_p)$ such that $\bar{g} = \bar{f}$, we would have $\bar{\rho}_g$ equivalent to $\bar{\rho}_f$, by the Cebotarev density and the Brauer-Nesbitt theorem. In other words, for any such g the representation ρ_g is a deformation of $\bar{\rho}_f$.

This deformation problem has the same formulation as that occurs in algebraic geometry. Using Schlessinger's criterion, Mazur [**3**] showed that there is an universal deformation; namely there exists a complete noetherian local $\mathbf{Z}_p$ algebra $\mathcal{R}(\bar{\rho})$, a representation:

$$\rho_{\mathrm{univ}} : G_{\mathbf{Q},S} \to \mathrm{GL}_2(\mathcal{R}(\bar{\rho}))$$

with the property that for any representation ρ as above, there exists a unique $\mathbf{Z}_p$ algebra homomorphism $\varphi_\rho : \mathcal{R}(\bar{\rho}) \to \mathbf{Z}_p$, such that ρ is equivalent to $\varphi_\rho \circ \rho_{\mathrm{univ}}$. In other words, $\mathrm{Spec}(\mathcal{R}(\bar{\rho}))$ served as a parameter space for the deformations of $\bar{\rho}$.

$\mathcal{R}(\bar{\rho})$ is "big". In fact, one can show [**3**] that the Krull dimension of $\mathcal{R}(\bar{\rho})/p\mathcal{R}(\bar{\rho})$ is ≥ 3. If $\bar{\rho}$ comes from the p-torsion of an elliptic curve over $\mathbf{Q}$, then using a theorem of Flach [**1**], one can show in a lot of cases that $\mathcal{R}(\bar{\rho})$ is isomorphic to a formal power series ring $\mathbf{Z}_p[[T_1, T_2, T_3]]$.

Furthermore, one can restrict the set of deformations allowed. For instance, recall that a representation:

$$\rho : G_{\mathbf{Q},S} \to \mathrm{GL}_2(A),\ A \text{ being } \mathbf{Z}_p\text{-algebra}$$

is said to be *ordinary* at p, if when restricted to the decomposition group at p, the representation can be put into a upper triangular form, with the rank one quotient being unramified (at p). It's an important theorem that, if f is p-ordinary, then ρ_f, and hence $\bar{\rho}_f$ is ordinary at p. In this case, one can restrict to deformations of $\bar{\rho}_f$ which are ordinary at p. Such ordinary deformations can be parametrised by universal ordinary deformation ring $\mathcal{R}(\bar{\rho})^o$, which is a quotient of $\mathcal{R}(\bar{f})$:

$$\mathcal{R}(\bar{\rho}) \to \mathcal{R}(\bar{\rho})^o$$

(geometrically, a closed locus of $\mathrm{Spec}(\mathcal{R}(\bar{\rho}))$ such that, the representation:

$$\rho^o_{\mathrm{univ}} : G_{\mathbf{Q},S} \to \mathrm{GL}_2(\mathcal{R}(\bar{\rho})^o)$$

obtained by composing ρ_{univ} with the above homomorphism, is ordinary at p, and has the required universal property.

Galois representations with "big" coefficient rings were historically first constructed in conjunction with Hida's theory. Indeed suppose that f is p-ordinary, and to avoid technical details, suppose $N = 1$. Recall the ordinary component of the p-adic Hecke algebra:

$$\mathbf{T}^{\mathrm{ord}} = h_k^{\mathrm{ord}}(\Gamma_1(p^\infty), \mathbf{Z}_p) = \varprojlim h_k^{\mathrm{ord}}(\Gamma_1(p^r), \mathbf{Z}_p)$$

Then a p-ordinary eigenform g as above corresponds to an algebra homomorphism $\varphi_g : \mathbf{T}^{\mathrm{ord}} \to \mathbf{Z}_p$. In other words, $\mathrm{Spec}(\mathbf{T}^{\mathrm{ord}})$ served as a universal parameter space that interpolates p-ordinary eigenform.

For our purpose, it suffices to study the local structure of $\mathrm{Spec}(\mathbf{T}^{\mathrm{ord}})$ around the closed point of characteristic p corresponding to the residual eigenform $\bar{f}$: it corresponds to $\varphi_f \pmod p$:

$$\mathbf{T}^{\mathrm{ord}} \to \mathbf{Z}_p \to \mathbf{F}_p$$

Let m be the kernel. Then m gives a closed point of $\mathrm{Spec}(\mathbf{T}^{\mathrm{ord}})$. Let $\mathbf{T}^{\mathrm{ord}}_{\mathrm{m}}$ be the completion of $\mathbf{T}^{\mathrm{ord}}$ at m. Then $\mathrm{Spec}(\mathbf{T}^{\mathrm{ord}}_{\mathrm{m}})$ can be interpreted as a local deformation space, which parametrises p-ordinary eigenform g, such that $\bar{g} = \bar{f}$.

Hida [**2**], and later Mazur-Wiles [**5**], showed that the association:

$$g \to \rho_g$$

can be interpolated; they constructed a Galois representation

$$\rho^{\mathrm{ord}}_m : G_{\mathbf{Q},S} \to \mathrm{GL}_2(\mathbf{T}^{\mathrm{ord}}_{\mathrm{m}})$$

such that if g is an p-ordinary eigenform such that $\bar{g} = \bar{f}$ as above, then ρ_g is equivalent to $\varphi_g \circ \rho^{\mathrm{ord}}_m$.

Moreover, Mazur and Wiles have shown that, the representation ρ^{ord}_m is ordinary at p in the sense above. Thus, $\mathrm{Spec}(\mathbf{T}^{\mathrm{ord}}_{\mathrm{m}})$ can also be regarded as the universal ordinary modular deformation space for the residual representation $\bar{\rho}_f$.

Now from the universal property of the deformation ring $\mathcal{R}(\rho)^o$, there exists an algebra homomorphism:

$$\mathcal{R}(\bar{\rho})^o \to \mathbf{T}_{\mathrm{m}}^{\mathrm{ord}}$$

such that ρ_m^{ord} is equivalent to the composition of ρ_{univ}^o with this homomorphism.

It's not difficult to show that this is surjective. Furthermore, Gouvêa and Mazur have conjectured that it is an isomorphism (in the general case where $N \neq 1$ the conjecture needs to be modified somewhat). This would be a precise way of saying that all the deformations of the residual representation $\bar{\rho}$, which is assumed to be modular, are also modular.

A variant of this conjectured isomorphism figured prominently in the work of Wiles, Taylor-Wiles on the Shimura-Taniyama conjecture. In turn, their methods and results can be used to show, in many cases, that the above conjecture of Gouvêa and Mazur is true. In fact, generalisation to rank 2 Siegel modular form, still in the ordinary case, has already been considered, *c.f.* Tilouine's lectures.

For a discussion of the non-ordinary setting, see [**4**].

References

[1] M.Flach *A finiteness theorem for the symmetric square of an elliptic curve* Invent. Math. **109** (1992), no.2, 307-327.

[2] H.Hida, *Galois representations into $GL_2(\mathbf{Z}_p[[X]])$ attached to ordinary cusp forms*, Invent. Math. **85** (1986), 545-613.

[3] B.Mazur *Deforming Galois Representations* Galois groups over **Q**, (Berkeley, CA, 1987), 385-437.

[4] B.Mazur *an "infinite fern" in the universal deformation space of Galois representations*, Collect. Math. **48**, 1997, 155-193.

[5] B.Mazur, A.Wiles *On p-adic analytic families of Galois representations* Compositio Math. **59**, no.2, 231-264.

Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA 02138

E-mail address: mok@math.harvard.edu