Mar 17. 2006. Friday Peter Schneider (4)

Lecture 4. p-adic Banach spaces.

Parameter (ξ, ζ)

$\xi \in X^*(T) = X_*(T') \subseteq X_*(G')(K)$
dominant.

$\zeta \in T'(K) = \mathrm{Hom}(\Lambda, K^\times) \subseteq G'(K)$ s.t. $\zeta \in T'_{\xi, \mathrm{norm}}$

$\downarrow$

$B_{\xi, \zeta}$: unitary Banach space rep'n of G.

$\boxed{\text{Conjecture}}$. $B_{\xi, \zeta} \neq 0$

$T'_{\xi, \mathrm{norm}} := \mathrm{val}^{-1}(V_\xi^{\mathrm{norm}})$

$V_\xi^{\mathrm{norm}} = \{ z \in V_{\mathbb{R}} : z^{\mathrm{dom}} \leq \eta_{\mathbb{Q}_p} + \xi_{\mathbb{Q}_p} \}$

$V_\xi = \{ z \in V_{\mathbb{R}} : (z + \eta_{\mathbb{Q}_p})^{\mathrm{dom}} \leq \eta_{\mathbb{Q}_p} + \xi_{\mathbb{Q}_p} \}$

$FIC_K :=$ Category of filtered K-Iso crystals

$\underline{D} = (\underset{\substack{\uparrow \\ \text{fin. dim} \\ K\text{-v.sp.}}}{D}, \underset{\substack{\uparrow \\ K\text{-linear} \\ \text{action}}}{\varphi}, \underset{\substack{\uparrow \\ \text{decreasing} \\ \text{filtration}}}{\mathrm{Fil}^\cdot D})$

$FIC_K^{\mathrm{adm}} \subseteq FIC_K$: full subcategory of (weakly) admissible objects.

Faltings | Totaro : FIC_K^{adm} is a <u>neutral</u> <u>Tannakian</u> category

Colmez | Fontaine : $FIC_K^{adm} \cong \mathrm{Rep}_K^{crys}(\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p))$.

$\boxed{A}$

$G = GL_{d+1}(\mathbb{Q}_p)$

$\cup$

P lower triangular Borel

$\cup$

T diagonal matrices

$U_0 = GL_{d+1}(\mathbb{Z}_p)$. $T_0 = T \cap U_0$

$$T/T_0 = \Lambda \ni \lambda_i := \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & p & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} T_0$$

i-th spot

$$V_{\mathbb{R}} = \mathrm{Hom}(\Lambda, \mathbb{R}) \xrightarrow{\cong} \mathbb{R}^{d+1}$$

$$z \longmapsto (z(\lambda_1), \dots, z(\lambda_{d+1}))$$

$$X^*(T) \otimes \mathbb{R} \xrightarrow{\cong} V_{\mathbb{R}} \xrightarrow{\cong} \mathbb{R}^{d+1}$$

$$\xi = \left[\begin{pmatrix} g_1 & & 0 \\ & \ddots & \\ 0 & & g_{d+1} \end{pmatrix} \mapsto \prod_i g_i^{a_i} \right] \longmapsto (a_1, \dots, a_{d+1})$$

dominant $a_1 \leq \dots \leq a_{d+1}$

<u>Note</u>: $\tilde{\eta}_{\mathbb{Q}_p} = (0, 1, \dots, d) = \eta_{\mathbb{Q}_p} + \frac{1}{2}(d, \dots, d)$, W-invariant

$$\eta_{\mathbb{Q}_p} = \frac{1}{2}(-d, -(d-2), \dots, d-2, d)$$

Use on T' the coordinates $T'(K) = \mathrm{Hom}(\Lambda, K^\times) \longrightarrow (K^\times)^{d+1}$

dual torus to T

$$\xi \longmapsto (\xi(\lambda_1), p\xi(\lambda_2), \dots, p^d \xi(\lambda_{d+1}))$$

$$V_\xi = \{ z \in V_{\mathbb{R}} : (z + \overline{\eta}_{\mathbb{Q}_p})^{dom} \leq \overline{\eta}_{\mathbb{Q}_p} + \xi_{\mathbb{Q}_p} \}$$

Fact : Under these coordinates

$T'_\xi = \mathrm{val}^{-1}(V_\xi)$ corresponds to

$$(\omega_p(\varsigma_1), \cdots, \omega_p(\varsigma_{d+1}))^{\mathrm{dom}} \leq (a_1, a_2+1, \cdots, a_{d+1}+d).$$

moreover : 1) dom means rearranging i in increasing order.

2) $(z_1, \cdots, z_{d+1}) \leq (z'_1, \cdots, z'_{d+1})$

if $z_{d+1} \leq z'_{d+1}$, $z_d + z_{d+1} \leq z'_d + z$

$$\vdots$$

$$z_2 + \cdots + z_d + z_{d+1} \leq z'_2 + \cdots + z'_d + z'_{d+1}$$

$$z_1 + z_2 + \cdots + z_{d+1} = z'_1 + z'_2 + \cdots + z'_{d+1}$$

$\underline{D}$ in FIC_K of dim $d+1$.

$\rightsquigarrow$ • $\varphi \in G_{d+1}(K) = G'(K)$

• $\mathrm{Fil}^{\cdot}D$ has a type, type$(\underline{D})$ which can be viewed as a dominant element in $X^*(T)$

$\boxed{\text{Thm}}$ For $\xi \in X^*(T)$ dominant and $\varsigma \in T'(K)$, the followings are equivalent.

i) $\varsigma \in T'_\xi = \mathrm{val}^{-1}(V_\xi)$

ii) There is a $\underline{D}$ in $\mathrm{FIC}_K^{\mathrm{adm}}$ s.t. $\varphi^{ss} = \varsigma$ and type$(\underline{D}) = \xi$.

Using the Fontaine Functor

$$\begin{array}{ccc} \text{Our parameter } (\xi, \varsigma) & \longmapsto & \overset{\text{(collection)}}{\text{Family}} \text{ of crystalline Galois rep'n} \\ \big\downarrow & & \updownarrow ? \\ B_{\xi,\varsigma} & \rightsquigarrow & \text{Family of all topological irred. quotients} \end{array}$$

$\boxed{B}$ G: general (split)

$\mathrm{REP}_K(G') :=$ Category of K-rational rep'n of G'

$\searrow$ <u>neutral</u> <u>Tannakien</u> <u>Category</u>

Let's look at a general pair

$(v, b) \in X_*(G') \times G'(K)$.

We have a functor

$$I_{(v,b)} : \mathrm{REP}_K(G') \longrightarrow \mathrm{FIC}_K$$

$$(\rho, E) \longmapsto \left(E, \rho(b), \begin{array}{l}\text{weight filtration} \\ \text{of co-character } \rho\circ v\end{array}\right)$$

<u>Def</u>. (v, b) is called <u>admissible</u>, if $I_{(v,b)}$ has image in FIC_K^{adm}

(v, b) admissible

$$\Rightarrow \begin{tikzcd} \mathrm{REP}_K(G') \arrow[r, "I_{(v,b)}"] \arrow[dr, dashed, "\substack{\text{faithful} \\ \text{tensor} \\ \text{functor}}"'] & \mathrm{FIC}_K^{adm} \arrow[d, "\cong"] \\ & \mathrm{Rep}_K^{crys}(\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)) \end{tikzcd}$$

$$\xRightarrow[\text{Formalism}]{\text{Tannakian}} \quad \gamma_{(v,b)} : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \longrightarrow G'(\overline{K})$$

determined up to conjugation on the target.

<u>Problems</u>: 1) Functoriality forces us to use $T'_{\xi, \mathrm{norm}}$

$\rightarrow$ need $p^{\frac{1}{2}} \in K$

2) $\eta_{\mathbb{Q}_p}$ <u>not</u> <u>integral</u>

Thm. Let $\xi \in X_*(T')$ dominant and $\delta \in T'(K)$, assume that $\eta_{\mathbb{Q}_p}$ is integral, then the following are equivalent.

(i) $\delta \in T'_{\xi,\mathrm{norm}}$

(ii) There is an admissible pair (v, b) such that $v \in G'(K)$-orbit of $\xi \cdot \eta_{\mathbb{Q}_p}$ and $b^{ss} = \delta$

What if $\eta_{\mathbb{Q}_p}$ is not integral ?

$\boxed{\text{Thm}}$ Everything remains the same if we work with filtered Bocrystals whose filtrations are indexed by $\frac{1}{2}\mathbb{Z}$ and

if we replace $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ by its unique nontrivial central extension.

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)_{(2)} \longrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to 0.$$

Provided K is big enough. (any element of $\mathbb{Q}_p^*$ is a square in K^*)

$$\varepsilon_2^2 = \varepsilon \qquad \begin{tikzcd} \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)_{(2)} \arrow[r] \arrow[d, dashed, "\varepsilon_2"'] & \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \arrow[r] \arrow[d, "\varepsilon \text{ . cyclotomic char}"] & 0 \\ K^* \arrow[r, "(\cdot)^2"] & K^* \end{tikzcd}$$