

Mar 17, 2006. Friday Peter Schneider (F) 17.

Lecture 4. p -adic Banach spaces.

Parameter (ξ, ζ)

$$\xi \in X^*(T) = X_*(T') \subseteq X_*(G')(K)$$

dominant.

$$\zeta \in T(K) = \text{Hom}(\Lambda, K^*) \subseteq G'(K) \text{ s.t. } \zeta \in T'_{\xi, \text{norm}}$$

↓

$B_{\xi, \zeta}$: unitary Banach space rep'n of G .

[Conjecture] $B_{\xi, \zeta} \neq 0$

$$T'_{\xi, \text{norm}} = \text{val}^{-1}(V_{\xi}^{\text{norm}})$$

$$V_{\xi}^{\text{norm}} = \left\{ z \in V_{\mathbb{R}} : z^{\text{dom}} \leq \eta_{\mathbb{Q}_p} + \frac{1}{\xi} \right\}$$

$$V_{\xi} = \left\{ z \in V_{\mathbb{R}} : (z + \eta_{\mathbb{Q}_p})^{\text{dom}} \leq \eta_{\mathbb{Q}_p} + \frac{1}{\xi} \right\}$$

FIC_K = Category of filtered K -iso crystals

$$\underline{D} = (D, \varphi, \text{Fil}^* D)$$

fm. dim
K-v. sp

K-linear
action

decreasing
filtration

$\text{FIC}_K^{\text{adm}} \subseteq \text{FIC}_K$: Full subcategory of (weakly) admissible objects.

Faltings / Totaro : FIC_K^{adm} is a neutral Tannakian category

Colmez / Fontaine : $FIC_K^{adm} \cong Rep_K^{crys} (Gal(\mathbb{Q}_p/\mathbb{Q}_p))$

[A]

$G = GL_{d+1}(\mathbb{Q}_p)$

U

P lower triangular Borel

U

T diagonal matrices

$U_0 = GL_{d+1}(\mathbb{Z}_p)$, $T_0 = T \cap U_0$

$T/T_0 = \Lambda \ni \lambda_i = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & p & \\ & & & \ddots \\ & & & & 1 \end{pmatrix} T_0$
2-th spot

$V_{\mathbb{R}} = Hom(\Lambda, \mathbb{R}) \xrightarrow{\cong} \mathbb{R}^{d+1}$

$z \mapsto (z(\lambda_1), \dots, z(\lambda_{d+1}))$

$X^*(T) \otimes \mathbb{R} \xrightarrow{\cong} V_{\mathbb{R}} \xrightarrow{\cong} \mathbb{R}^{d+1}$

$\xi = \left[\begin{pmatrix} g_1 & & & \\ & \ddots & & \\ & & g_{d+1} & \\ & & & \ddots \end{pmatrix} \mapsto \prod_i g_i^{w_i} \right] \mapsto (Q_1, \dots, Q_{d+1})$

dominant $Q_1 \leq \dots \leq Q_{d+1}$

Note : $\tilde{\eta}_{\mathbb{Q}_p} = (0, 1, \dots, d) = \eta_{\mathbb{Q}_p} + \frac{1}{2}(d, \dots, d)$, W-invariant.

$\eta_{\mathbb{Q}_p} = \frac{1}{2}(-d, -(d-2), \dots, d-2, d)$

Use on T' the coordinates $T(K) = Hom(\Lambda, K^{\times}) \rightarrow (K^{\times})^{d+1}$
dual torus $\cong T$

$\xi \mapsto (s(\lambda_1), p s(\lambda_2), \dots, p^d s(\lambda_{d+1}))$

$V_{\xi} = \{ z \in V_{\mathbb{R}} : (z + \tilde{\eta}_{\mathbb{Q}_p})^{dom} \in \tilde{\eta}_{\mathbb{Q}_p} + \sum_{\mathbb{Q}_p} \}$

Fact: Under these coordinates

$T'_\xi = \text{val}^{-1}(V_\xi)$ corresponds to

$$(\omega_p(\xi_1), \dots, \omega_p(\xi_{d+1}))^{\text{dom}} \leq (a_1, a_2+1, \dots, a_{d+1}+d)$$

moreover: 1) dom means rearranging i in increasing order.

2) $(z_1, \dots, z_{d+1}) \leq (z'_1, \dots, z'_{d+1})$

if $z_{d+1} \leq z'_{d+1}$, $z_d + z_{d+1} \leq z'_d + z'_{d+1}$

$z_2 + \dots + z_d + z_{d+1} \leq z'_2 + \dots + z'_d + z'_{d+1}$

$z_1 + z_2 + \dots + z_{d+1} = z'_1 + z'_2 + \dots + z'_{d+1}$

\underline{D} in FIC_k of dim $d+1$.

$\rightarrow \circ \varphi \in G_{d+1}(K) = G'(K)$

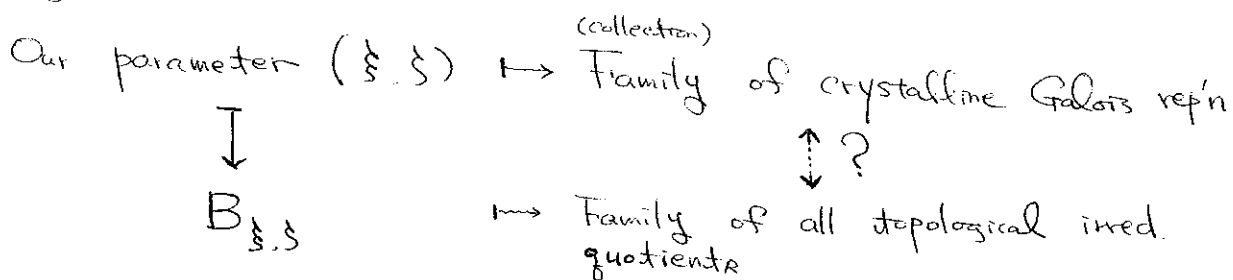
\circ Fil \underline{D} has a type, $\text{type}(\underline{D})$ which can be viewed as a dominant element in $X^*(T)$

Thm For $\xi \in X^*(T)$ dominant and $\zeta \in T'(K)$, the followings are equivalent.

i) $\zeta \in T'_\xi = \text{val}^{-1}(V_\xi)$

ii) There is a \underline{D} in $\text{FIC}_k^{\text{adm}}$ st. $\varphi^{\text{ss}} = \zeta$ and $\text{type}(\underline{D}) = \xi$

Using the Fontaine Functor



B G : general (split)

$\text{REP}_K(G') :=$ Category of K -rational rep'n of G'

↳ neutral Tannakien Category

Let's look at a general pair

$$(v, b) \in X_*(G') \times G'(K)$$

We have a functor

$$I_{(v,b)}: \text{REP}_K(G') \longrightarrow \text{FIC}_K$$

$$(P, E) \longmapsto (E, P(b), \text{weight filtration of co-character } P \circ v)$$

Def. (v, b) is called admissible, if $I_{(v,b)}$ has image in $\text{FIC}_K^{\text{adm}}$

(v, b) admissible

$$\Rightarrow \text{REP}_K(G') \xrightarrow{I_{(v,b)}} \text{FIC}_K^{\text{adm}}$$

faithful
tensor
functor

\cong

$$\text{Rep}_K^{\text{crys}}(\text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p))$$

Tannakian
Formalism

$$\gamma_{(v,b)} = \text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \longrightarrow G'(\bar{K})$$

determined up to conjugation on the target.

Problems: 1) Functoriality forces us to use $T_{\xi, \text{norm}}'$
→ need $p^{\frac{1}{2}} \in K$

2) $\eta_{\mathbb{Q}_p}$ not integral

Thm. Let $\xi \in X_*(T')$ dominant and $\zeta \in T'(K)$,

assume that $\eta_{\mathbb{Q}_p}$ is integral, then the following are equivalent.

- (i) $\zeta \in T'_{\xi, \text{norm}}$
- (ii) There is an admissible pair (v, b) such that $v \in G'(K)$ -orbit of $\xi \cdot \eta_{\mathbb{Q}_p}$ and $b^{ss} = \zeta$

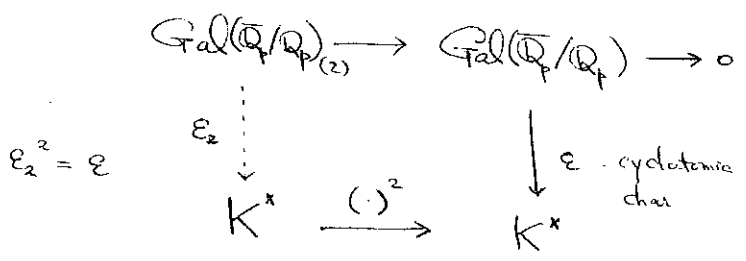
What if $\eta_{\mathbb{Q}_p}$ is not integral?

Thm

Everything remains the same if we work with filtered Bocrystals whose filtrations are indexed by $\frac{1}{2}\mathbb{Z}$ and if we replace $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ by its unique nontrivial central extension.

$$1 \rightarrow \{\pm 1\} \rightarrow \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)_{(2)} \rightarrow \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \rightarrow 0.$$

Provided K is big enough, (any element of \mathbb{Q}_p^* is a square in K^*)



$$\varepsilon_2^2 = \varepsilon$$