3. 15. joint w/ J. Teitel, C. Breuil.

$L = \mathbb{Q}_p$. $\qquad w_p$: add. valn.

G : $\mathbb{Q}_p$-pts of a split conn. reduc. gp $/\mathbb{Q}_p$.

$\cup\!|$

$P = TN$. Borel. $\quad < \begin{array}{l} T : \text{max tor} \\ N : \text{unip rad.} \end{array}$

$W = N(T)/T$ Weyl gp.

$U_0 :=$ "good" max'l. cpt subgp.

$T_0 := T \cap U_0$.

$\Lambda := T/T_0$. free abel gp

$$X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T) \xrightarrow{\cong} \Lambda.$$
$$v \longmapsto v(p)T_0$$

Classical unramif Langlands functoriality

$K = \overline{K}$ char 0. $\qquad$ Fix $p^{\frac{1}{2}} \in K$.

G'/K conn. Langlands dual gp.

$\cup\!|$

T' torus dual to T. $\quad$, $T'(K) = \mathrm{Hom}(\Lambda, K^\times)$.

$$X^*(T) \cong X_*(T') = \mathrm{Hom}(\mathbb{G}_m, \mathrm{Hom}(\Lambda, \mathbb{Z}) \otimes \mathbb{G}_m)$$
$$\chi \longmapsto [\, a \mapsto w_p \circ \chi \otimes a \,]$$

Satake-Hecke alg

$\mathcal{H}(G, 1_{U_0}) :=$ all loc. const fcns w/ cpt supp.

$$\psi : U_0 \backslash G / U_0 \to K.$$

K-alg wrt $\displaystyle \psi_1 * \psi_2(h) = \sum_{g \in G/U_0} \psi_1(g)\psi_2(g^{-1}h)$.

"univ. mod over $\mathcal{H}(G, 1_{U_0})$"

left $G \circlearrowright \boxed{\mathrm{ind}_{U_0}^G(1)} := $ loc. const fcns w/ cpt supp. $f : G/U_0 \to K$.

right

$G \longleftrightarrow \mathcal{H}(G, 1_{U_0})$

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idea Fix the action of $\mathcal{H}(G, 1_{U_0})$ to get. smaller rep.
Need to know str. of ↑

Satake isom

$$S^{norm}: \mathcal{H}(G, 1_{U_0}) \longrightarrow K[\Lambda]$$
$$\psi \longmapsto \sum_{\lambda=\lambda(t)} \delta^{-\frac{1}{2}}(t) \sum_{n\in N/N_0} \psi(tu)\,\lambda.$$

induces an isom of K-alg
$$\mathcal{H}(G, 1_{U_0}) \xrightarrow{\cong} K[\Lambda]^W. \qquad (W \text{ acts on } T, T_0, \Lambda)$$

Aside (Peter: ferial in $\hat{G}$, not in G. no need.
mazur: should we pin down $\hat{G}$ more (and data) to gain feriality?)

δ = modulus char. of P.
$$\delta(t) = |\det(ad(t), \mathrm{Lie}\,N)|_p^{-1} \subseteq p^{\mathbb{Z}} \subseteq \mathbb{Q}^\times \subseteq K^\times.$$
$\delta \in T'(K)$.
$p^{\frac{1}{2}} \in K \rightsquigarrow \delta^{\frac{1}{2}} \in T'(K)$.

note: One can drop the normaliz^n by $\delta^{-\frac{1}{2}}$ gets
corresp. isom. w/ W-action changed by certain cocycle.
(is defined always).
(But to gain feriality, $\delta^{-\frac{1}{2}}$ is necessary).

note: $K[\Lambda] = \mathcal{O}_{alg}(T')$, $K[\Lambda]^W = \mathcal{O}_{alg}(W\backslash T')$

$$\mathrm{Max}(\mathcal{H}(G, 1_{U_0})) = (W\backslash T')(K) = \left\{\begin{array}{l}\text{set of s.s. conj.}\\ \text{classes in } G'(K)\end{array}\right\}$$
$\Downarrow$
ζ.

give image of Frob_p ↑ $\qquad \updownarrow$

$$\left\{\begin{array}{l}\text{isom classes of unram.}\\ \text{s.s. Weil gp param.}\\ W_{\mathbb{Q}_p} \to G'(K)\end{array}\right\}$$

$\{V_\zeta\}_\zeta$ ↑ unram. Langlands feriality ↙

Then we can specialize
$$H_\zeta := \mathrm{ind}_{U_0}^G(1) \otimes_{\mathcal{H}(G, 1_{U_0})} K_\zeta.$$

H_ξ is a fin. length smooth G-rep. and has a unique irred quot V_ξ.

We bring in an irred $\mathbb{Q}_p$-rat'l rep (ρ, E) of G, corresp. to a highest wt $\xi \in X^*(T)$.

$K/\mathbb{Q}_p$ fin.

$\boxed{\mathcal{H}(G, \rho|u_0)} :=$ ~~loc. const.~~ cpt supp. ftn $\psi : G \to \mathrm{End}_K(E)$ satisfying $\psi(u_1 g u_2) = \rho(u_1) \circ \psi(g) \circ \rho(u_2)$.

$\mathrm{ind}_{u_0}^G(\rho|u_0)$ with G acting on the left and $\mathcal{H}(G, \rho|u_0)$ acting on the right (not smooth rep, but ...)

Fix a U_0-invar norm $\| \ \|$ on E.

$\rightsquigarrow$ operator norm on $\mathrm{End}_K(E)$.

$\rightsquigarrow$ sup-norms on $\mathcal{H}(G, \rho|u_0)$ and on $\mathrm{ind}_{u_0}^G(\rho|u_0)$.

complete.

$\mathcal{H}(G, \rho|u_0) \rightsquigarrow B(G, \rho)$ K-Banach alg

$\mathrm{ind}_{u_0}^G(\rho|u_0) \rightsquigarrow B_{u_0}^G(\rho)$ K-Banach space w/
- cont. isometric action of G
- cont. action of $B(G, \rho)$.

Strategy: specialize $\mathrm{ind}_{u_0}^G(-)$ via $B(G, \rho)$, get smaller rep.

note: $\mathcal{H}(G, \rho|u_0) \cong \mathcal{H}(G, 1_{u_0}) \xrightarrow[\text{Satake}]{\cong} K[\Lambda]^W = \mathcal{O}_{alg}(W\backslash T')$

$\psi \cdot \rho \mapsfrom \psi$

point: The two norms on these are very different.

Again $p^{\frac{1}{2}} \in K$.

Define norm $\| \ \|_\xi$ on $K[\Lambda]$ by

where w (depending on λ) is chosen s.t. $w\lambda$ is anti-dominant

$$\Big\| \sum_{\lambda \in \Lambda} c_\lambda \lambda \Big\|_\xi := \sup_{\lambda = \lambda(t)} \Big| \delta^{\frac{1}{2}}({}^w\lambda)\, p^{w_p(\xi({}^w t))} c_\lambda \Big|$$

valn

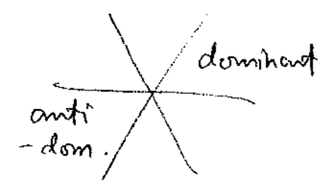
Prop 1: $\left(\mathcal{H}(G, \rho|_{U_0}), \text{sup-norm}\right) \cong \left(K[\Lambda]^W, \| \ \|_{\xi}\right)$.

Further notation.

val: $T'(K) = \mathrm{Hom}(\Lambda, K^\times) \xrightarrow{\omega_p} \mathrm{Hom}(\Lambda, \mathbb{R}) =: V_{\mathbb{R}}$ root space

$$V_{\mathbb{R}} \ni \eta_{\mathbb{Q}_p}, \xi_{\mathbb{Q}_p} \longleftrightarrow \xi \in X^*(T) \otimes \mathbb{R} \xrightarrow{\cong} V_{\mathbb{R}}, \quad \chi \mapsto \omega_p \circ \chi$$

$\eta_{\mathbb{Q}_p} = \frac{1}{2}$ sum of pos. roots.



W acts on $V_{\mathbb{R}}$.

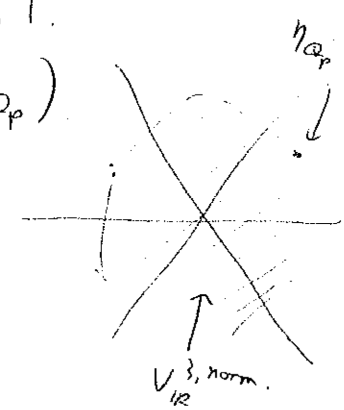
$\exists$ a partial order $\leq$ on $V_{\mathbb{R}}$.
(def'd by pos. roots).

GL_3

$V_{\mathbb{R}} \ni z \mapsto z^{dom} :=$ dom. pt in Wz.

Define $\boxed{V_{\mathbb{R}}^{\xi, norm}} := \{ z \in V_{\mathbb{R}} : z^{dom} \leq \eta_{\mathbb{Q}_p} + \xi_{\mathbb{Q}_p} \}$.
$=$ convex hull of ${}^W(\eta_{\mathbb{Q}_p} + \xi_{\mathbb{Q}_p})$

e.g.



$\boxed{T'_{\xi, norm}} := \mathrm{val}^{-1}(V_{\mathbb{R}}^{\xi, norm})$.

Prop 2

(i) $T'_{\xi, norm}$ is an affinoid subdomain in T', W-invariant.

(ii) $\mathcal{B}(G, \rho|_{U_0}) \cong \mathcal{O}(W \backslash T'_{\xi, norm})$.

parameter (ξ, ζ) $\quad$ (ξ : a highest wt. $\zeta \in T'_{\xi, norm} \subseteq T'$.

We can define the "specialization"

$$\mathcal{B}_{\xi, \zeta} := K_\zeta \hat{\otimes} \mathcal{B}^G_{U_0}(\rho)$$

is a unitary Banach space rep of G.

Big problem: Is $\mathcal{B}_{\xi,\zeta} \neq 0$? $\quad$ could be easier, but completion