Lecture 7 Eigenvarieties at some non tempered points and Selmer groups (with J. Bellaïche)

1) A conjecture of Bloch-Kato

$E/\mathbb{Q}$ q. imaginary, π cusp. aut. rep of $GL_n(\mathbb{A}_E)$

s.t. i) $\pi^\vee \simeq \pi^c |\ |^{-1}$

ii) π_∞ algebraic regular ($\pi_\infty |\ |^{-\frac{1}{2}}$)

Assume $\exists\ \rho : G_E \longrightarrow GL_n(\bar{\mathbb{Q}}_p)$ comp LLC outside p, pure ...

i) $\Rightarrow$ $L(\pi, s) = \mathcal{E}(\pi, s)\, L(\pi, -s)$

Conj (BK). $\mathrm{ord}_{s=0} L(\pi, s) = \dim H^1_f(E, \rho)$

(weaker), $\mathcal{E}(\pi, 0) = -1 \Rightarrow H^1_f(E, \rho) \neq 0$

Assumptions i) $p = v\bar{v}$ splits in E, π_p unram.

ii) π_w ramified $\Rightarrow$ w splits in E (not poss. n odd, forget it to simplify)

iii) $4 \nmid n$ and $0, -1$ are not HTW of $\rho|_{D_v}$

Theorem 1 Rep $(n+2)$ + $AC(\pi)$ $\Rightarrow$ Sign conj.

Theorem 2 Assume V + $\begin{cases} \text{i) } H^1_f(E, \rho(-1)) = H^1_f(E, \mathrm{ad}\,\rho)^{\mathrm{auto}\perp} = 0 \\ \text{ii) } \Lambda^i \rho \text{ irred } \forall \text{ all } 1 \leq i \leq n \\ \text{iii) } \pi_v \text{ has a reg. refinement, non critical} \end{cases}$

Then for some explicit $\mathcal{E} \ni x$, $\dim T_x(\mathcal{E}) \leq r\left(n + \frac{r+1}{2}\right) + 1$

eigenv. of $U(n+2)$

$\dim V \subseteq H^1_f(E, \rho)$

subspace we can construct from $(\mathcal{E}, x)$.

Coroll: $r = 1 \Rightarrow \mathcal{E}$ smooth at x.

Rmks - Sign conject. known $n=1$ Rubin (Greenberg?), (BC, Annales ENS 2004)
$n=2$ coming from mod. forms, Nekovar's, Sk-Urb. same cases.
parity

- concentrate on Thm 2, | nothing is conjectural $n=1$
good choices of E, start from mod. forms | Rep (4) is mainly missing

2) The conjecture $AC(\pi)$

. $4 \nmid n \underset{\text{Hasse's pr.}}{\Longrightarrow} \exists\ U(n+2)_{/\mathbb{Q}}$ q-spl. all finite places, real points compact.

. Endoscopic functoriality $\quad {}^L U(2) \times {}^L U(n) \longrightarrow {}^L U(n+2)$

Arthur conjectures that π should transfer to an "A-packet" of

rep $\Pi = \{\pi'\}$ of $U(n+2)$.

whose Galois rep. (assuming Rep $(n+2)$) is $1 + \omega + \rho$ (not tempered but discrete.

$* \to$ not tempered, discrete.

. Interested in a special element "base element"

Let $\pi_0 = \pi_\infty' \otimes \pi_v$ the (obvious) rep of $U(n+2)(\mathbb{A})$ attached to $1+\omega+\rho$ by LLC. (use $0,-1$ not weight π_∞)

Then

Conj $(AC(\pi))$ $\quad \pi_0 \hookrightarrow L^2(U(n+2)(\mathbb{Q}) \backslash U(n+2)(\mathbb{A}), \mathbb{C})$ if $\varepsilon(\pi, \iota) = -1$

img. *Actually iff, and if it is the case, multiplicity should be 1)

* know $n=1$ (Rogawski), other cases? ($n=2$?)

* In Bell's lecture, he will maybe describe Π when $n=1$. (Rogawski)

Rough idea where "the Selmer ele^{ts} comes from"

along the lines of Ribet's work on converse of Herbrand's thm.

idea to use these Π when $\varepsilon(\Pi, 0) = -1$ due to several people (Harder). metric.

Bellaïche thesis (2001) $n = 1$.

i) Deform $1 + \omega + \rho$ to some ρ' att. to a stable temp. (in.) form on $U(3)$.

(congruences, level raising)

ii) Lattice argument, $\exists$ factors, at least a serious issue that R.L not true

to produce 1 by ρ, we must show that 1 by ω (e.g.) does not appear.

show it is "f" and use that $\mathcal{O}_E^\times$ finite. (Kummer theory).

Skinner-Urban $\sim GSp_4$ use families to produce deformation arg. as in Wiles-MC.

$\rightarrow$ class const. in char 0.

iii) discover cannot use ord. family, π_v not tempered.

(4) Proof thm 2

$\varepsilon(\Pi, 0) = -1$, $AC(\Pi) \Rightarrow \exists \pi_0 \hookrightarrow L^2(U(n+2)(\mathbb{Q}) \backslash U(n+2)(\mathbb{A}), \mathbb{C})$

choose a minimal level (type BK) for π_0.

- K_w max hyp. sp. $\forall$ non split w
- BK at split places, enflated if mon (possible...)
- K_p = iwahori

$H = \mathcal{A}_p \otimes$ spherical at all unram places $\neq p$

$\rightarrow$ red. eigenvariety $\mathcal{E}$, Rep $(n+2)$ as in lect. 6 $T : G_E \longrightarrow \mathcal{O}(\mathcal{E})$.

- choose a point for π_0 $\Leftrightarrow$ chose a p-refinement of $\pi_{0,p}$

$$(1, \underbrace{\varphi_1, \ldots, \varphi_n}_{\text{regular}}, p^{-1}) \quad \begin{cases} \text{accessible as 1 preceeds } p^{-1} \\ \text{parity of } \rho \end{cases}$$

- $T : G_E \to A$, $A = \mathcal{O}_{X,x}$, $m = m_{X,x}$, $k := A/m$, $K = \mathrm{Frac}\, A$

  $T \bmod m = 1 + \omega + \rho$ <u>m. free</u>

- analyse the galois rep. quite carefully, in the spirit of MW, W.

we want to compute $S := \dfrac{A[G]}{\mathrm{Ker}\, T}$

- <u>Lecture 5</u>

$$S = \begin{array}{cccc} & \rho & \omega & 1 \\ & \multicolumn{3}{l}{\left(\begin{array}{ccc} M_n(A) & A^n_{\omega\rho} & A^n_{1,\rho} \\ A^n_{\rho\omega} & A & A_{1\omega} \\ A^n_{\rho,1} & A_{\omega 1} & A \end{array}\right)} \end{array} \subset M_{2+n}(K)$$

A_{ij} fract. ideals
+ chasles rel.
+ $A_{ij} A_{ji} \subset m$, $j \neq i$

<u>Moreover</u> $\tau : g \mapsto c^{-1} g^{-1} c\, \omega(g)$
factors through $\tau : S \to S$ anti involution, $\tau \circlearrowright \{1, \omega, \rho\}$

Lemma about <u>idemp.</u> (in the lifting), τ induces isom $A_{i,j} \xrightarrow{\sim} A_{\tau(j),\tau(i)}$

<u>Aim</u> <u>compute $A_{i,j}$'s</u>.

Ⓐ <u>Red. locus</u>

<u>lemma</u> The total red. locus of T is m.

<u>pf</u>: Use <u>lecture</u> 6, reducible point $T \bmod m = 1 + \rho + \omega$
in this order, refinement <u>is interwall</u>. $(1, \phi_1, \dots, \phi_n, \rho^{-1})$
compute <u>permutation</u> σ, weight $k_1 < k_2 < \dots < \underset{k_i}{\overset{\shortparallel}{1}} < \underset{k_{i+1}}{\overset{\shortparallel}{0}} < k_{i+2} < \dots < k_{n+2}$

$$\sigma \left| \begin{array}{l} 1 \to i+1 \\ 2 \to 1 \\ 3 \to 2 \\ \vdots \\ i \to i-1 \\ i+1 \to i+2 \\ \dots \to n+2 \end{array} \right.$$

it is <u>a cycle</u>.

Hence by lect. 6, $\forall\, I \subset \mathfrak{m}$ cofinite length
I_{tot}

i) $T \bmod \mathfrak{m} = \tilde{\hat{1}}_{A/I} + \tilde{\hat{\omega}}_{A/I} + \mathrm{tr}\, \tilde{\hat{\rho}}_{A/I}$ (know triangulation).

all crystalline deformations

but $H^1_f(E, \mathrm{ad}\,\rho) = 0 = H^1_f(E, \mathbb{Q}_p) \Rightarrow$ all are cst. $\Rightarrow$ F_i's are cst.
hyp. — finiteness of ideal class group.

ii) weights are cst if one is.

but $\mathcal{O}_{X,x}$ is generated over $\mathcal{O}_{W,x(x)}$ by T_ℓ's U_p's $\Rightarrow I = \mathfrak{m} \Rightarrow I_{tot} = \mathfrak{m}$.

Corollary $\mathfrak{m} = A_{1\omega} A_{\omega 1} + A_{1\rho} A_{\rho 1} + A_{\omega\rho} A_{\rho\omega}$.

B) Ext comp.

$$\mathrm{Ext}_{\mathrm{Sel}}(\rho_i, \rho_j) \xrightarrow{\sim} \mathrm{Hom}\left(A_{ij}/A_{ik}A_{kj}, k\right)$$
$$\cap$$
$$\mathrm{Ext}^1_{k[G]}(\rho_i, \rho_j)$$

Remark

$i \backslash j$	1	ω	ρ
1		f 0	f ?
ω	self d.c. ≤ 1		
ρ	$\leq$	*	*

(i,j) box about $\mathrm{Ext}_{\mathrm{Sel}}(\rho_i, \rho_j)$

i) All theses Ext fall into f-outside p pair

ii) at p

iii) duality.

iv) dim are bounded by

i) Outside p, BK type at some argument.

ii) ~~At~~ p, 2 places $v \to 1$, $\bar{v} \to \rho^{-1}$ — reasons as in lect. 6

+ duality

iii) properties of τ.
abstract

v) dim $\quad H^1_f(E, \mathbb{Q}_p(1)) = \mathcal{O}_E^\times \otimes \mathbb{Q}_p = 0$

$H^1_f(E, \mathbb{Q}(-1)) = 0$ (Soulé, MW) $\Rightarrow$ ass. 1 by $\tilde{\omega}'$

$H^1_f(E, \rho(1)) = 0$ by assumption

local comp. using fundam. exact sequence + $\left|\begin{array}{l} \text{wt} \neq 0, -1 \\ \varphi \neq 1, p^{-1} \end{array}\right.$ (punts)

$\Rightarrow \underline{\dim H^1(E, \rho(1)) \leq n}$.

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c) <u>End argument</u>.

- <u>NAK</u> $\Rightarrow A_{1\omega} = A_{1\rho} A_{\rho\omega}$, $\quad A_{1\omega} A_{\omega 1} \subset A_{1\rho} A_{\rho 1}$
- <u>duality</u> $\Rightarrow \quad A_{\rho\omega} A_{\omega\rho} = A_{1\rho} A_{\rho 1}$
- $\Rightarrow \quad I_{tot} = m = A_{1\rho} A_{\rho 1}$
- $A_{1\omega} A_{\omega\rho} = A_{1\rho} A_{\rho\omega} A_{\omega\rho} \subset m A_{1\rho} \overset{NAK}{\Rightarrow}$

  ming. of $A_{1,\rho} = \dim \mathrm{Ext}_{Sen}(1, \rho) =$ max number of indp. ext. $= r$ of 1 by ρ we can find in lattices of K^{n+2}.

- $$A_{\rho,1} = \sum_{i=1}^{n} \underset{\underset{k}{\cap}}{A f_i} + A_{\rho\omega} A_{\omega 1}$$

  $$\left.\begin{array}{l} \\ \text{and} \quad A_{\omega 1} = \underset{\underset{K}{\cap}}{h} A + A_{\omega\rho} A_{\rho 1} \end{array}\right\} \overset{NAK}{\Longrightarrow} A_{\rho,1} = \sum_{i=1}^{n} A f_i + \underset{\substack{\cup \\ \lambda A_{1\rho} \\ \lambda \in k^*}}{h A_{\rho\omega}}.$$

$$\Rightarrow \quad A_{\rho\omega} A_{\omega 1} = h A_{\rho\omega} + \underbrace{A_{\rho\omega} A_{\omega\rho}}_{\subset m} A_{\rho 1}$$

already $\Rightarrow \quad m$ has $\leq r(n+r)$

<u>but</u> $\quad m = A_{\rho 1}\left(\sum^{n} A f_i\right) + h \lambda A^{2} \ldots \Rightarrow \boxed{\dim T_{\mathfrak{m}}(\mathcal{E}) \leq r(1 + \ldots)}$

<u>Rmks</u>

* find examples to know! (computable)
* link with p-adic L-functions
* does not follows from Main Conj.