Lecture ⑤ Residually multiplicity free Pseudocharacters (jt wk with Joël Bellaïche) | April 5 G. Chenevier

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Need for pseudocharacters : construction of Galois rep associated to p-adic eigenform is indirect

① Definitions

A commutative ring, $d \geq 1$, $d!$ invertible in A.

R a A-algebra.

Def (Wiles $n=2$, Taylor, wk. Procesi) An pseudoc. of dim d on R is an A-linear map $T : R \to A$

such that
i) $T(1) = d$
ii) $T(xy) = T(yx)$ $\forall x, y \in R$
iii) $S_{d+1}(T) = 0$ on R^{d+1}

expl $c = (i_1 \cdots i_s) \in \mathfrak{S}_n$; $S_n(T)(x_1, \dots, x_n) = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\sigma} \prod_{\sigma = c_1 \cdots c_r} T^{c_i}(x_1, \dots, x_n)$

$S_n(T)$, $n \geq 1$. $T^c(x_1, \dots, x_n) := T(x_{i_1} \cdots x_{i_s})$

Rks i) $S_n(T)$ multilinear sym. ; $R = A[G]$. T determined by $T|_G$, $S_{d+1}(T)|_G$

ii) Cayley Hamilton id

$d!$ invertible $\Rightarrow$ form abstract $P_{x,T} = X^d - T(x) X^{d-1} + \cdots$

$\Rightarrow T$ is the full polarization of CH. identity $x^d - T(x) x^{d-1} + \frac{T(x^2) - T(x)^2}{2} X^{d-2} + \cdots$

e.g $d = 2$

$R = M_2(A)$ $0 = P_{x+y,T}(x+y) - P_{x,T}(x) - P_{y,T}(y) = xy + yx - T(x) y - x T(y) + T(x)T(y) - T$

$T = \mathrm{tr}$. $\forall x, y \in R$ $T = 0$.

$$\boxed{S_3(T)(x,y,z) = 2T(\text{—} z)} = 0)$$

in general (Procesi) $T(\underbrace{CH_T(x_1 \cdots x_d)}_{\text{full pol. of CH id.}} x_{d+1}) = d! \, S_{d+1}(T)(x_1, \dots, x_{d+1})$ (*)

(d any)

...tion $T : R \to A$ is C.H if $P_{x,T}(x) = 0$ $\forall x \in R$.
$\in R$

Ex i) $\ker T = \{x,\ T(xy) = 0 \ \forall y\}$ 2-sided ideal of R
then $T : R/_{\ker T} \to A$ is CH by $\circledast$
ii) others!

$\rightsquigarrow$ study of CH-algebras (Procesi's point of view)

② Appl. to eigenvarieties

reduce it if not already

Recall G, choice of $\mathcal{X}$. $\rightsquigarrow$ $\mathcal{E} \supset Z$ a Zariski dense set of cl. points

as known! $\begin{cases} \text{Assume that } \forall z \in Z,\ \exists \rho_z : G_{E,S} \longrightarrow GL_n(\bar{\mathbb{Q}}_p) \text{ cont, ss trace} \\ \forall v \notin S,\ tr(\rho_z(Frob_v)) = T_v(z). \end{cases}$

Prop. $\exists!\ T : G_{E,S} \to \mathcal{O}(\xi)^{\leq 1}$ cont., dim d, such that $\forall z \in Z$, $T_z = tr\ \rho_z$.

Pf. i) let $\tilde{\mathcal{X}} = \overline{\text{im}(\mathcal{X} \to \mathcal{O}(\xi))}$ it is compact. ($\mathcal{E}$ reduced, $\tilde{\mathcal{X}} \subset \mathcal{O}(\mathcal{E})^{\leq 1}$

ii) $T' : G \to \prod_{z \in Z} \mathbb{C}_p$ $(g \mapsto (Tr(\rho_z)))$ pseudo c.

$\uparrow$. $\text{im}\, T' \subset \tilde{\mathcal{X}}$ hence lifts
Chebotarev

$\dashrightarrow \tilde{\mathcal{X}} \subset \mathcal{O}(\mathcal{E})$

$\subset$ compact ~~closed~~, homeomorphic to its image.

Now $\forall x \in \mathcal{E}$, evaluation T_x, use.

Theorem (Taylor) if $T : R \to k$ alg closed char 0, then T is the trace of a ss. rep. of dim dim T.

$\rightsquigarrow$ gives a ρ_x $\forall x$.

Rk. Need more to study families, applying this to generic points of $\mathcal{E}$ gives something. Not enough in general however.

Residually multiplicity free pseudo c.

A local henselian, residue field $A/\mathfrak{m} = k$, $d!$ invt in A

$T : R \to A$ d-dim pseudo car.

(applies to classical Hecke alg. $/\mathbb{Z}_p$, Hida families, def rings, $\mathcal{O}_x^{rig}$, $x \in \mathcal{E}$)

Assume $T \bmod \mathfrak{m} : R \otimes k \to k = \sum_{i=1}^{r} \mathrm{tr}\, \bar{\rho}_i$, where $\bar{\rho}_i : R \twoheadrightarrow M_{d_i}(k)$
(MF). 2 by 2 non isom.

Theorem (BCh) Assume T is MF, CH.

i) $\exists$ $A \subset B$ commutative ring, $A_{ij} \subset B$ sub A module
such that $A_{ii} = A$, $A_{ij} A_{jk} \subset A_{ik}$, $A_{ij} A_{ji} \subset \mathfrak{m}$ if $j \neq i$.

and $R \underset{\text{trace} = T}{\hookrightarrow} M_d(B)$ with image $\bigoplus_{i,j} M_{d_i d_j}(A_{ij})$

Assume Moreover $\mathrm{Ker}\, T = 0$:

ii) if A is reduced, noetherian, $K = \mathrm{Frac}\, A$, then we can take $B = K$ and A_{ij} are frac. ideals.

iii) if A is factorial, we may take $B = A$

Rq i) $r = 1 \Rightarrow R \underset{T = \mathrm{tr}}{=} M_d(A)$ (Nyssen, Rouquier in the $\ker T = 0$ case)

ii) the abstract A_{ij} are well defined indep. of B.

Pf (Sketch of (i))

(a) $\mathrm{Rad}\, R = \ker (T \otimes k)$
hence $R / \mathrm{Rad}\, R \simeq \prod_{i=1}^{r} M_{d_i}(k)$

(b) hensel $\Rightarrow$ $\exists$ $1 = e_1 + \dots + e_r$ ON family of idempotents lifting.

(c) claim $T|_{e_i R e_i}$ is a pseudo c. dim d_i which is CH.

(d) if we know $r = 1$, $e_i R e_i = M_{d_i}(A)$ and Morita concludes.

$d=1$ $R/_{\mathrm{rad}R} \simeq M_d(k)$. Hensel + Bk. $\Rightarrow$ lifts E_{ij}

$\rightsquigarrow$ again, we reduce to $d=r=1$. But then $R=A$ obvious. ■

④ Consequences of the structure thm (same ass, T CH, MF)

(I) Reducibility ideals

prop (i) $\exists$ smallest ideal $I_{tot} \subset A$ such that

$I_{tot} \subset J \subset m$ iff $T \bmod J = T_1 + \cdots + T_r$, $T_j : R/_{JR} \to A/_J$ pseudo ch. dir. lifting $\mathrm{tr}\,\bar{\rho}_i$.

(ii) We have $I_{tot} = \sum_{j \neq i} A_{ij} A_{ji}$

(II) Module theory

Let $I_{tot} \subset J \subset m$, $T \bmod J = \sum T_j$, $T_j = \mathrm{tr}\,\rho_j$
(use thm) $\rho_j : R \twoheadrightarrow M_{d_j}(A/J)$.

Prop

$$\boxed{\mathrm{Ext}_{R/JR}(\rho_j, \rho_i) \xrightarrow{\sim} \mathrm{Hom}_{A/JA}\left(A_{ij} \Big/ \sum_{k \neq i,j} A_{ik}A_{kj},\ A/J\right)}$$

"Pf". Set $M_j = \bigoplus_i A_{ij}^{d_i}$ (some column of R)
it is a projective R-module.

• Say J, as above.
claim $0 \to \bigoplus_{i \neq j} \left(\frac{A_{ij}}{JA_{ij}}\right)^{d_i} \to M_j/JM \to A/J^{d_j} \to 0$ S-module.
$\simeq \rho_j$

apply $\mathrm{Hom}_{R/JR}(-, \rho_i)$ and play with Morita and idempotents.

NB: formulas for Ext^n also $n \geq 2$, less useful in applications.

Examples

$$A = k[[x,y]]/xy \subset \tilde{A} \simeq k[[x]] \times k[[y]]$$

$$R = \begin{pmatrix} A & (x,1)\tilde{A} \\ (y,y)\tilde{A} & A \end{pmatrix}, \quad T = k. \quad \subset M_2(\tilde{A})$$

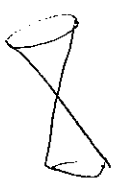
ex $\Rightarrow$ no true representation in $M_2(A)$

red locus $= (x,1)\tilde{A}\,(y,1)\tilde{A} = m$

ext groups, $\chi_1 \neq \chi_2$ two characters $R \xrightarrow{\chi_1,\chi_2} k$

$\mathrm{Ext}_{R\otimes k}(\chi_1,\chi_2)$ has dim 2 $=$ nb generators of $\tilde{A}$ as A-module.

(b) ~~A = k[[...]]~~ ~~A = k[[z]]~~



$A = k[[x,y,z]]/(xy - z^2)$

$$R = \begin{pmatrix} A & (y,z)A \\ (\frac{z}{x})A + A & A \end{pmatrix}$$

$(y,z)(\frac{z}{x}A + A) = m$ again.

Ext's have both dim 2