Lecture ④ Eigenvarieties for definite unitary groups | March 24 | G. Chenevier ①

Back to lecture ③ | - $n=2$ Stevens "o.c. modular symbols GL_2" (preprint 95')
Buzzard "p-adic and fams quat. algebras"
- any n, independently - Ash-Stevens " ——— Gl_n" (prep. 2000)
- Ch. " Crelle's 2004.

Today Coleman's "p-adic families..."
Buzzard's "eigenvarieties"

Ⓐ Unitary groups

1. k field, E/k étale $d^\circ 2$, $\Delta_{/E}$ central simple algebra rk n^2 + $*: \Delta \longrightarrow \Delta$ anti k-linear involution inducing the non trivial $E \longrightarrow E$

$\leadsto$ group functor on k-algebras A by

$$G(A) = \{ x \in (\Delta \otimes_k A)^\times,\ xx^* = 1 \}$$

e.g. * $E = k \times k$, $\Delta \simeq \Delta_1 \times \Delta_2$, $\Delta_2 \overset{*}{\simeq} \Delta_1^{opp}$, $G \xrightarrow[\text{choice of } i]{\sim} \Delta_i^*$, e.g. $Gl_n(k)$ arises this way

* $\Delta = M_n(E)$, $*$ is the adjunction for a hermitian form

when $E \neq k \times k$, we talk about unitary groups.

2. $k = \mathbb{Q}$, $E/\mathbb{Q}$ quadr. imaginary field, then

i) if $p = u\bar{u}$ splits in E, $G(\mathbb{Q}_p) \xrightarrow[u]{\sim} \Delta_{E_u}^\times$, and for a.all p $\xrightarrow[u]{\sim} Gl_n(\mathbb{Q}_p)$

ii) p not split, unitary group

iii) $G(\mathbb{R}) \xrightarrow[\iota]{\sim} U(a,b)$ unit. gp. signature (a,b) $a+b=n$.

fix $E \xrightarrow{\iota} \mathbb{C}$

Today's assumptions i) Fix $p = u\bar{u}$ split, and u, such that $G(\mathbb{Q}_p) \xrightarrow[u]{\sim} Gl_n(\mathbb{Q}_p)$

ii) signature $(a,b) = (n,0)$ or $(0,n)$.

Rmk. plenty of them: $\exists$ Hasse's principle for unitary gps utomorphic forms

$$\mathcal{A} = L^2\left(G(\mathbb{Q})\backslash G(\mathbb{A}), \mathbb{C} \right) \underset{\text{thm.}}{=} \overset{\text{top.} \perp}{\bigoplus} m(\pi)\, \pi \quad , m(\pi) \neq 0, \text{ finite}$$

π automorphic rep. of G. irreducible

right translation of $G(\mathbb{A})$; each π is discrete, algebraic, cuspidal.

Main facts

* finiteness class number: $G(\mathbb{A}_f) = \coprod_{i=1}^{h} G(\mathbb{Q})\, x_i\, K$ $\forall K$ a comp. op. subgroup of $G(\mathbb{A}_f)$
* $G(\mathbb{Q}) \hookrightarrow G(\mathbb{A}_f)$ is discrete, hence arithmetic subgroups of G, i.e. $G(\mathbb{Q}) \cap K$, are all finite.

Rk. $G(\mathbb{Q})\backslash G(\mathbb{A}) / G(\mathbb{R}) \times K$ finite, however arithmetically very rich.

* Fix K c.o. subgroup, $k \in \mathbb{Z}^{n,+}$. By $G(\mathbb{R}) \hookrightarrow Gl_n(\mathbb{C})$, may view $V_k(\mathbb{C})$ as a rep. of $G(\mathbb{R})$, we get all of the continuous ones this way.

Def $f \in \mathcal{A}$ is an aut. form of wt k, level K, if $K.f = f$ and $G(\mathbb{R}).f$ is a finite sum of V_k^*.

© Hecke-Iwahori algebra

Recall $I \subset Gl_n(\mathbb{Q}_p)$, $M = \langle I, U^+ \rangle$

proposition $\mathbb{Z}[I \backslash M / I]$ is a commutative ring: $[IuI][Iu'I] = [Iuu'I]$ (Schneider-St.)

call it $\mathcal{H}_p$.

Let π be an irreducible smooth unramified representation of $Gl_n(\mathbb{Q}_p)$ / $\mathbb{C}$ such that $\pi^I \neq 0$.

$\pi \rightsquigarrow$ Langlands class of geom Frob. $\in Gl_n(\mathbb{C})$
semi-simple element.
$L(\pi)$

$$\{\text{gen. Eigenspaces of } \mathcal{H}_p \circlearrowleft \pi^I\} \xleftrightarrow[\text{img}]{\text{natural}} \{\text{ordering of eigenvalues of } L(\pi)\} =: \{p\text{-refine}$$
$(\phi_1 \cdots \phi_n)$

- ~~[illegible]~~, the image are the orderings such that $\phi_i = p\,\phi_j \Rightarrow i < j$ (check?)

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ex: • generic case, $n!$ ordering
- π = trivial, unique one $(p^{+\frac{(n-1)}{2}}, \ldots, p^{-\frac{n-1}{2}})$.

| Call these orderings the "accessible" p-refinements of π.

D) The Eigenvariety

Fix - $K = K_p \times K^p$ with $K_p \underset{\text{u}}{\hookrightarrow} Gl_n(\mathbb{Q}_p)$ iwahori subgroup.
- $\mathcal{H}$ a commutative Hecke algebra$_{/\mathbb{Z}}$ $\supset \mathcal{H}_p$ at p, the unramified Hecke alg. at al. all places.

Eigenforms $f \neq 0 \in \mathcal{A}$ give rise to a ring homomorphism $\psi_f : \mathcal{H} \to \mathbb{C}$.

Fix $\bar{\mathbb{Q}} \nearrow \mathbb{C}$, $\searrow \bar{\mathbb{Q}}_p$, it is a fact (algebraicity) that $\psi_f(\mathcal{H})$ falls in $\bar{\mathbb{Q}}$ hence may be viewed as $\bar{\mathcal{O}}_p$ valued, and it makes sense to talk about congruences.

Let $\mathcal{W} = \mathrm{Hom}(T(\mathbb{Z}_p), \mathbb{G}_m) \supset \mathbb{Z}^{n+1}$ as before.

Theorem (Ch (indep Emerton))

There exists a rigid analytic space $\mathcal{E}$ over $\mathbb{Q}_p$ which is separated, nested, equidim n equipped with
(a) A Ring hom. $\psi : \mathcal{H} \longrightarrow \mathcal{O}(\mathcal{E})^{\leq 1}$
(b) A finite map $(\kappa, (F_i)_{i=1}^{n}) : \mathcal{E} \longrightarrow \mathcal{W} \times \mathbb{G}_m^n$
with κ locally finite and surjective $(\mathcal{E} \to \mathcal{W})$
(c) A Zariski-dense subset $Z \subset \mathcal{E}(\bar{\mathbb{Q}}_p)$, such that.

i) If $z \in Z$, $\psi_z = \psi_{f_z}$ for a eigen form f_z of weight $k(z)$ and level K.
$k(z) \in \mathbb{Z}^{n+1}$
Any such form appears this way.

...over, $\left(p^{\mathrm{rank}_n(z)} F_n(z), \dots, p^{k_1(z)} F_1(z) \right)$ is the p-refinement of f_z associated to $\Psi_{f_z} | \mathcal{H}_p$.

ii) $x, y \in X(\overline{\mathbb{Q}}_p)$ are equal iff $\Psi_x = \Psi_y$

iii) If $x \in X(\overline{\mathbb{Q}}_p)$ is such that $k(x) = (k_1 \geq \dots \geq k_n) \in \mathbb{Z}^{n,+}$
and $v\left(F_1(x) F_2(x)^2 \cdots F_{n-1}(x)^{n-1} \right) < 1 + \mathrm{Min}_{i=1}^{n-1} (k_i - k_{i+1})$, then $x \in Z$.

Ⓔ <u>Proof</u>

General strategy closed to Coleman's one.

ⓐ $\mathbb{Q}_p$-<u>model for automorphic forms</u>

Let F be the functor $\{M\text{-modules}\} \longrightarrow \{\mathcal{H}\text{-modules}\}$

$$F(V) := \left\{ f : G(\mathbb{Q}) \backslash G(\mathbb{A}_f) \longrightarrow V, \ f(xu) = u_p^{-1} f(x) \ \ \forall u \in K_p \right\}$$

$G(\mathbb{Q}) \backslash G(\mathbb{A}_f) / K = \{x_1, \dots, x_h\}$

$$\downarrow \wr \quad f \mapsto (f(x_i))$$

$$\prod_{i=1}^{h} V^{\Gamma_i}, \qquad \Gamma_i = K \cap x_i^{-1} G(\mathbb{Q}) x_i \text{ finite group.}$$

<u>Fact</u>: $S_k(K) \xrightarrow{\sim} F(V_k^*)$ (similar to the way we associate p-adic Hecke char. to complex ones)

$S_k(K)$: $\mathcal{H}$-module of forms wt k, level K

ⓑ <u>$S_k(K) \hookrightarrow$ p-adic aut. forms</u>

$$F(V_{-k}) \overset{\mathcal{H}}{\hookrightarrow} F(\mathcal{V}_k) = \mathcal{S}_k(K)$$ defined in previous lecture.

Same argument $\Rightarrow$ $[IU^{+}I]$ $\circlearrowright$ compactly (explain Banach norm).
small slope forms are classical.

$\Omega \subset \mathcal{W}$, $r \geq r_\Omega$ as before, it makes sense to define

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$$\mathcal{S}_{\Omega,r}(K) := F(\mathcal{V}_{\Omega,r}) \quad \begin{cases} \text{PON-able } \mathcal{O}(\Omega,r)\text{-module} \\ \mathcal{H} \text{ acts by continuous end. norm} \leq 1 \\ [IuI],\ u \in U^{++} \text{ are compact.} \end{cases}$$

(D) | $U_p = [I(1 \cdots p^{h_{n,1}}) I]$
$\rightsquigarrow$ $\det(1 - TU_p)$ well defined element in $\mathcal{O}(\mathcal{W} \times \mathbb{G}_m)$

whose evaluation at $k \in \mathbb{Z}^{k,+} \hookrightarrow \mathcal{W}$ is $\det(1 - TU_p \mid \mathcal{S}_k(K))$.

(use the fact that F commutes with any base change)

(E) Construct from this the spectral variety of $U_p \subset \mathcal{W} \times \mathbb{G}_m$, and above this the eigenvariety $\mathcal{E}$ as in Kevin's lectures (use Kevin's covering)

NB: - Locally, $k : \mathcal{E} \to \mathcal{W}$ has all the properties of $B \subset M_n(A)$
A open aff. of $\mathcal{W}$.

- We can give ~~precise~~ explicit radius for families (factorisation of $CPS(U_p)$) in terms of the class number h. (extending Wan's approach, Kevin's also).