5 April 11. 2006. Tuesday. Kevin Buzzard. 1:00pm

15th Lecture

$\overline{\mathbb{Q}}_p \cong \mathbb{C}$

$\ell \neq p$.

$\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_2(\overline{\mathbb{Q}}_p)$

f : cuspidal eigen form $\xrightarrow{\text{Deligne}}$

$\rho_f|_{D_\ell} : G_{\mathbb{Q}_\ell} \longrightarrow GL_2(\overline{\mathbb{Q}}_p)$

$\pi_{f,\ell} \circlearrowleft GL_2(\mathbb{Q}_\ell)$ $\xleftrightarrow[\text{LANGLANDS}]{\text{LOCAL}}$ 2-dim Weil-Deligne gp = Deligne-Langlands conjecture (up normalization)

Grothendieck + F-semi-simplification: Weil-Deligne

This has concrete conseq.

e.g. tells you $\rho_f|_{D_\ell}^{ss}$ even if $\ell \mid$ level of f.

e.g. it tells you that if $\ell \nmid$ level then $\rho_f(\mathrm{Frob}_\ell)$ has char poly

$$X^2 - a_\ell X + \ell^{k-1} \chi(\ell)$$

Another concrete conseq. :

If $E/\mathbb{Q}$ is an elliptic curve & f is the associated new form, then $\mathrm{cond}(E) = $ level of f.

Funny special case of Local Langlands $\quad \pi \longleftrightarrow (\rho, N)$

π can be 1-dimensional.

$$\rho = \text{twist of} \begin{pmatrix} |\cdot|^{\frac{1}{2}} & 0 \\ 0 & |\cdot|^{-\frac{1}{2}} \end{pmatrix} \quad \& \quad N = 0$$

$$\updownarrow$$

$$\pi = 1\text{-dim}$$

These π's show up as subquotients of $I(\chi_1, \chi_2)$, $\chi_1/\chi_2 = |\cdot|^{\pm 1}$

The other subquotient is a twist of St & local Langlands matches up the twist of Steinberg with $\boxed{\text{a twist of} \begin{pmatrix} |\cdot|^{\frac{1}{2}} & 0 \\ 0 & |\cdot|^{-\frac{1}{2}} \end{pmatrix}. N \neq 0}$

"More than one possible N"

$\Longleftrightarrow$ reducibility of the induced rep'n.

<u>Remark</u> about local-global for cusp forms

the 1-dim'l case never shows up

Thm if f is a cusp form & $\pi_{f,\ell}$ is 1-dim

then $f(g^{\mathbb{A}}) = \underset{\neq 0}{\underline{\text{cst}}} \times f \Rightarrow f = 0$

<u>On Galois Side</u>

We need to rule out the possibility that $\rho_f | D_\ell = \begin{pmatrix} \lambda(\ell \cdot \alpha) & * \\ 0 & \lambda(\alpha) \end{pmatrix}$

ramified $*$ should be o.k.

$* = 0$ should be impossible

$\alpha \in \overline{\mathbb{Q}}_p^\times$. $\lambda(\alpha)$ is the unram. char of D_ℓ sending arith Frob to α.

Looking at <u>det</u>

$\Rightarrow \ell \cdot \alpha^2 = \ell^{k-1} \cdot \chi_N(\ell)$

$\Rightarrow |\alpha| = \ell^{\frac{k-2}{2}}$

$\boxed{\begin{array}{l} a_\ell = \text{trace } \rho(\text{Frob}_\ell) \\ \quad \text{if } \ell \nmid \text{level} \end{array}}$

$\boxed{\begin{array}{l} * \neq 0 \\ \Rightarrow a_\ell = \alpha \end{array}}$

& $a_\ell = (\ell+1) \cdot \alpha \quad \therefore |a_\ell| = (\ell+1) \cdot \ell^{\frac{k-2}{2}} = \ell^{\frac{k}{2}} + \ell^{\frac{k-2}{2}}$

Weil bounds $\Rightarrow |a_\ell| \leq 2 \cdot \ell^{\frac{k-1}{2}}$ <u>contradiction</u>

Rmk: Harris-Taylor proved local Langlands for GL_n (l-adic field)

& Henniart gave a much simpler pf.

Harris + Taylor also prove local-global compatibility up semi-simplification.

G = unitary gp/$\mathbb{Q}$ ($\boxed{\text{self-dual}}$)

$G(\mathbb{Q}_p) \cong GL_n(\mathbb{Q}_p)$

ρ : rep of GL_n

Harris-Taylor also didn't match up N's

(ρ_1, N_1) (ρ_2, N_2) $\rho_1 \not\cong \rho_2$ ← Fixed by Taylor & Yoshida

Motivating Question

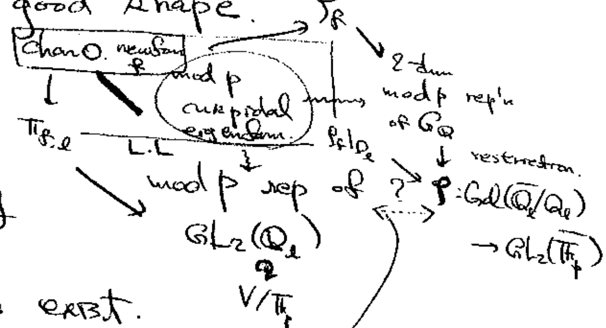
Can one formulate or prove other local langlands or local-global compatibility statements?

Another example when we're in good shape.

mod p rep'n at l. $p \neq l$.

Q) Is there a story relating

Some sort of picture does exist.



Vigneras (1990's GL_2, later GL_n)

Vigneras analysed "inductions" in char p.

$F/\mathbb{Q}_l$ finite ext'n

$$|\cdot| : F^\times \longrightarrow \overline{\mathbb{F}}_p^\times \qquad |l| = \frac{1}{l^{[F:\mathbb{Q}_l]}}$$

$|\text{uniformizer}| = \frac{1}{\#}$ # = # of residue field.

One has to fix a choice of $\sqrt{\#\text{ of residue field}}$ in $\overline{\mathbb{F}}_p$

If $\chi_1, \chi_2 : F^\times \longrightarrow \overline{\mathbb{F}}_p^\times$.

then one can define $I(\chi_1, \chi_2) = \left\{ \text{loc. const fns } f : GL_2(F) \to \overline{\mathbb{F}}_p \;\; f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \cdot g\right) = \chi_1(a)\chi_2(d)\sqrt{|a/d|}\, f(g) \right\}$

where $\sqrt{\left|\frac{a}{d}\right|} := \alpha^{v(d)-v(a)}$

<u>Results</u> $I(\chi_1, \chi_2)$ is irreducible if $\chi_1/\chi_2 \neq |\cdot|^{\pm 1}$

& In these cases the only isos are $I(\chi_1, \chi_2) \cong I(\chi_2, \chi_1)$

The rep ~~rep~~ rep'n $I(|\cdot|^{\frac{1}{2}}, |\cdot|^{-\frac{1}{2}})$ [This is the case that in char 0 had length 2 & Steinberg as sub & trivial 1 d. as quot.]

If # res field $\not\equiv -1$ mod p

then $I(|\cdot|^{\frac{1}{2}}, |\cdot|^{-\frac{1}{2}})$ has 2 J-H factors (one is 1-dim)

& # res field $\equiv +1$ mod p then it's direct sum of these factors.

If # of res. field $\equiv -1$ mod p then there are **3** J-H factors!

2 are 1-dim'l

the rep'n is <u>indecomposable</u>.

one 1-d sub

one 1-d quotient

& one "new" ∞-dim'l thing in middle, neither <u>a sub</u> nor <u>a quotient</u>

One can now attempt to put ~~together~~ a mod p local <u>Langlands</u>.

What one really <u>wants</u> is a mod p local-global compatibility.

<u>Here's the problem.</u>

If f is mod p cuspidal eigenform, of level $N\ell$, $p \nmid N\ell$, $\ell \nmid N$.

i.e. $f \in H^0(X_1(N\ell)/_{\mathbb{F}_p}, \omega^{\otimes k})$

& f is the mod p reduction of a char 0 newform of level $N\ell$.

It may happen that $\rho_f : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{F}}_p)$ maybe unramified at ℓ.

f (mod p) **lifts** to F (char 0 new form)

$\pi_{F,\ell} \cong$ unr. twist of Steinberg

& $\rho_F|D_\ell$ is ramified $\rho_F|D_\ell = \begin{pmatrix} \lambda(\ell\alpha) & * \\ 0 & \lambda(\alpha) \end{pmatrix}$ ramified.

but $\rho_f = \overline{\rho}_F$ is unramified at ℓ. because $* \equiv 0 \pmod p$

Rk. in global char 0 setting

$* = 0$ could not occur for global reasons

Weil bound.

e.g. $E/\mathbb{Q}$ ell. curve with split mult. red'n @ ℓ.

$$T_p(E) = \begin{pmatrix} \text{cyclo} & * \\ 0 & 1 \end{pmatrix}$$

& $* \neq 0$ because "if $* = 0$, then $a_\ell = \ell+1$"

$|a_\ell| \le 2\sqrt{\ell}$ (contr.)

Problem :

we certainly can have $a_\ell \equiv \ell+1 \mod p$.

e.g. (Ribet – Stein)

the elliptic curve

$$Y^2 + Y = X^3 + X^2 - 12X + 2$$

has conductor 3×47.

& associated wt level 3×47 MF is

$$f = q - 2q^2 + q^3 + 2 q^4 - 3q^5 - 2 q^6 - 3q^7.$$

so $a_3 = 1$ & E has split multi. red'n at 3

The j-invariant of E is $\dfrac{2^{12} \times 37^3}{3^7 \times 47}$

& Hence $E[7]$ is unramified at 3 & irreducible.

$\mathbb{Q}_\ell^\times / \langle a \rangle$. Conductor $(E[7]) = 47$

& there's a wt 2 modular form g of level 47.

s.t $\rho_g \mod 7 \underset{\text{Thm.}}{\cong} E[7]$.

q-exp'n of $g \equiv q + 5q^2 + 4q^3 + 2q^4 + 4q^5 + 6q^6 + 4q^7 + \cdots$ (mod 7)

$a_3 \equiv 1+3 \mod 7$.

The problem now is :

If f is the mod p^n modular form associated to E.

then how should we define $\pi_{f,(3)}$?

Naive classical data:

$\pi_{f,3}$ should be 1-dim. on $\rho_f | D_3 = \begin{pmatrix} \text{cyclo.} & 0 \\ 0 & 1 \end{pmatrix}$.

Problem: f (level 47×3) — lifts to F new form, G 3-old form

H = associated form of level 47.

& it would be nice, if $\pi_{f,3}$ was the reduction of $\underline{\pi_{F,3}}$ & of $\underline{\pi_{G,3}}$

$\pi_{F,3}$: Steinberg; $\pi_{G,3}$: principal series $I(\chi_1, \chi_2)$, $\chi_1/\chi_2 \neq |\cdot|^{\pm 1}$, $\chi_1/\chi_2 \equiv |\cdot|^{\pm 1} \bmod 7$

& the mod 7 reduction of $\pi_{G,3}$ is reducible,

with ss — it's $\overline{\pi}_{F,3}$ + 1-dim'l.

Best idea:

Let's define $\pi_{f,3}$ = Steinberg.

If f_0 is the mod 7 form of level 47.

then one can define $\pi_{f_0,3}$ to the rep'n that one gets

by mimicing the char 0 construction of last time

Then $\pi_{f_0,3}$ will be reducible.

It's the reduction of the irreducible $I(\chi_1, \chi_2)$