

G = reductive group. 12th lecture.

$\{ \text{Auto. rep'n for } G \} \cong \begin{pmatrix} \text{algebraic} \\ \text{cuspidal} \\ \text{auto. rep'n} \end{pmatrix} \hookrightarrow \text{Eigenvarieties?}$

IF $G = \text{GL}_2/\mathbb{Q}$, cuspidal alg auto. repns. \leftrightarrow twists of classical cusp new forms \hookrightarrow (Coleman-Mazur Eigen Curve) $\times W$
 + certain Maass forms \hookrightarrow conti even $(G_{\mathbb{Q}} \rightarrow \text{GL}_2(\mathbb{Q}))$ $\times W$
 (discrete set)

Unpublished work of Stevens and others
 - gives a family of p-adic L-fns on the eigen curve.

Other examples

1) $G = \text{Res}_{K/\mathbb{Q}} \text{GL}_1$ K : a number field.

G is an alg group over \mathbb{Q} s.t. if A is a \mathbb{Q} -alg
 $G(A) = (A \otimes_{\mathbb{Q}} K)^{\times}$

e.g. $G(\mathbb{C}) = (\mathbb{C} \otimes_{\mathbb{Q}} K)^{\times} \cong \mathbb{C}^{\times, s}$
 G is a torus

The automorphic rep's for G biject naturally with Größencharacters for A_K .

Recall if K/\mathbb{Q} finite,

$$A_K = \prod'_{\mathfrak{v}} K_{\mathfrak{v}}$$

\mathfrak{v} : a place of K

e.g. $A_{\mathbb{Q}}$ is the subgroup of $(\prod_{\mathfrak{p}} \mathbb{Q}_{\mathfrak{p}}) \times \mathbb{R}$ of elements $(x_{\mathfrak{v}})_{\mathfrak{v}}$ s.t. $x_{\mathfrak{p}} \in \mathbb{Z}_{\mathfrak{p}}$ for all but fin. many \mathfrak{p}

$$A_K = A_{\mathbb{Q}} \otimes_{\mathbb{Q}} K.$$

$$A_K^{\times} = \text{idèles}$$

A G.C. is cts gp homo.

$$K^x \backslash A_K^x \longrightarrow \mathbb{C}^x$$

usual
cx topology.

Facts about A_K^x

1) If $\mathcal{O} = \mathcal{O}_K$, then $\prod_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}}^x \subseteq A_K^x$
 \mathfrak{p} a prime

& if we define

$$K_{\infty} = K \otimes_{\mathbb{Q}} \mathbb{R} = \prod_{v \mid \infty} K_v \cong \mathbb{R}^{n_1} \times \mathbb{C}^{n_2} \quad (K_{\infty}^x)^{\circ}$$

$$K_{\infty}^x \cong (\mathbb{R}^x)^{n_1} \times (\mathbb{C}^x)^{n_2} \quad (\mathbb{R}_{>0})^{n_1} \times (\mathbb{C}^x)^{n_2} = \text{conn comp. of } 1$$

Then the following is exact.

$$\begin{matrix} \text{diagonally} \\ \circlearrowleft (\mathcal{O}_K^x) \xrightarrow{\text{diag}} K^x \times \prod_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}}^x \times (K_{\infty}^x)^{\circ} \longrightarrow A_K^x \longrightarrow \text{narrow class gp of } K \longrightarrow 0 \\ \text{finite} \quad \text{finite} \quad \text{infinite} \end{matrix}$$

(f-g) diag finite infinite (finite gp)

e.g. $K = \mathbb{Q}$: deduce

$$\mathbb{Q}^x \times \prod_{\mathfrak{p}} \mathbb{Z}_{\mathfrak{p}}^x \times \mathbb{R}_{>0} \cong A_{\mathbb{Q}}^x$$

\therefore a G.C. of \mathbb{Q} is a cts gp homo. $(\prod_{\mathfrak{p}} \mathbb{Z}_{\mathfrak{p}}^x) \times \mathbb{R}_{>0} \longrightarrow \mathbb{C}^x$

A cts gp homo. $\hat{\mathbb{Z}}^x \longrightarrow \mathbb{C}^x$ is a Dirichlet char. $\hat{\mathbb{Z}}^x$ is

Acts gp homo. $\mathbb{R}_{>0} \longrightarrow \mathbb{C}^x$ is a complex number s
 $x \mapsto x^s \quad s \in \mathbb{C}$

So for $K = \mathbb{Q}$, a G.C. is a pair (χ, s) ($\chi =$ a Dirich. char) $s \in \mathbb{C}$

Another fact about A_K^x .

$$\text{Gal}(K/K)^{ab} = \frac{K^x \backslash A_K^x}{(K_{\infty}^x)^{\circ}} \quad K^x \backslash A_K^x \sim \frac{\hat{\mathcal{O}}^x \cdot K_{\infty}^x}{\mathcal{O}^x}$$

(G.CFT) $\longrightarrow \quad A_K^x / \frac{K^x (K_{\infty}^x)^{\circ}}{K^x (K_{\infty}^x)^{\circ}}$

Say $\chi: K^x \backslash A_K^x \longrightarrow \mathbb{C}^x$ is a G.C.

$$\chi_{\infty} := \chi / (A_K^x)^{\circ}: (\mathbb{R}_{>0})^{n_1} \times (\mathbb{C}^x)^{n_2} \longrightarrow \mathbb{C}^x$$

Say X is algebraic

If χ_∞ is of the form

$$(x_1, \dots, x_m, z_1, \dots, z_n) \mapsto \prod_i X_i^{m_i} \times \prod_j Z_j^{r_j} \bar{z}_j^{s_j}$$

with $m_i, r_j, s_j \in \mathbb{Z}$

(Großchar) \cong (alg \mathbb{C} s)

Take observe that the set of all \mathbb{C} s naturally formed a complex analytic manifold.

$k = \mathbb{Q}$: only many copies of \mathbb{Q}

Any k : _____

Reason is $A_k^x \rightarrow \mathbb{R}_{>0}$

$$(a) \mapsto Na = \prod |a|$$

$|z|=4$

Product formula $\Rightarrow k^x \subseteq$ kernel

Get a $\mathbb{G}\mathbb{C}$ $\|\cdot\|$: $k^x / A_k^x \rightarrow \mathbb{P}^x$ \rightarrow cyclotomic character

Put 2 $\mathbb{G}\mathbb{C}$ s χ_1 & χ_2 into same $\mathbb{G}\mathbb{C}$

$$\Leftrightarrow \chi_1 / \chi_2 = \|\cdot\|^s, \text{ some } s \in \mathbb{C}$$

Take : If X is a $\mathbb{G}\mathbb{C}$.

he defined $L(X) \in \mathbb{C} \cup \{\infty\}$ st resulting fctn on \mathbb{R}, \mathbb{S} is meromorphic Riemann Surface

If X is an alg $\mathbb{G}\mathbb{C}$, then \exists compatible system of λ -adic \mathbb{G} -adics reps associated to X .

Idea of construction

$\chi_\infty = \chi / (k_\infty^x)^\circ$. Extend χ_∞ to all of k_∞^x in obvious way

$\chi \div \chi_\infty : A_k^x \rightarrow \mathbb{C}^x$ trivial on $(k_\infty^x)^\circ$ (non-trivial on k^x) (same formula)

Check: image of $\chi \div \chi_\infty$ is contained in a number field E .

If λ is a prime of E , $\lambda \nmid \ell$, then $(\chi \div \chi_\infty) / (k_\infty^x - (k_\infty^x)^\circ)^\times$

then $(\chi \div \chi_\infty) \Big|_{K^x} = K^x \longrightarrow E^x$
embedded diagonally

can be extended ctsly to a map

to a map $\psi: K_e^x \longrightarrow E_\lambda^x$
 " $(\pi \otimes \mathbb{Q}_e)^x$

$\chi \div \chi_\infty \div \psi$ is now trivial on K^x & $(K_\infty^x)^\circ$
 induces a map.

$\chi_\lambda = \text{Gal}(K/K)^{ab} \longrightarrow E_\lambda^x$

The χ_λ are all Hodge-Tate & unramified outside a finite set.

RR: If $\alpha: \text{Gal}(K/K) \longrightarrow \text{GL}(E_\lambda)$

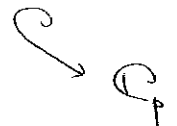
is H-T. & unram. outside a finite set

then arises from a GC in this way.

& Hence in particular compatible family

$(\text{Alg. GCs}) \xrightarrow{\text{same}} (p\text{-adic Theory})$

Fix embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$



$K^x \longrightarrow \mathbb{C}^x \xrightarrow{\tau \circ \sigma} \mathbb{C}^x$
 $\chi(\underline{a}) \div \chi_\infty(\underline{a}_\infty) = \psi(\underline{a}_\mathbb{Q})$

Now given χ , an alg. GC,

get $\chi_p = A_K^x / K^x \cdot (K_\infty^x)^\circ \longrightarrow \mathbb{C}_p^x$

but not all conti. maps.

$A_K^x / K^x \cdot (K_\infty^x)^\circ \longrightarrow \mathbb{C}_p^x$ arise in this way

ker χ
 \cup
 $\mathbb{C}_\mathbb{Q}^x$
 for all but fin. many \mathfrak{p}
 $\chi(\mathbb{Q}_\mathfrak{p}) \in \mathbb{C}_\mathfrak{p}^x$

Fix now a tame level

$\cup \subseteq \prod_{\substack{\mathfrak{p} \text{ fm} \\ \mathfrak{p} \neq p}} \mathbb{C}_\mathfrak{p}^x$ opt open

e.g. $K = \mathbb{Q}$
 if $N \geq 1, \mu_N$
 $\cup_N = \{a \in \prod_{\substack{\mathfrak{p} \\ \mathfrak{p} \neq p}} \mathbb{Z}_\mathfrak{p}^x, a \equiv 1 \pmod{N}\}$

If H is an abelian profinite gp. s.t H contains an open subgroup iso. to $(\mathbb{Z}_p)^n$ for some $n \geq 0$, then $\text{Hom}_{\text{cts gp}}(H, \mathbb{C}_p^\times)$ is naturally a rigid space over \mathbb{C}_p naturally \mathbb{C}_p -points of a rigid space over \mathbb{C}_p

Functor : rigid spaces \rightarrow gpa.

e.g. $H = \mathbb{Z}_p$: space is

$$\{w \in \mathbb{C}_p : |w-1| < 1\} \cong \text{open ball } B$$

If $H = \mathbb{Z}_p^n$: space is B^n

$H = \text{fm. gp.}$: space is discrete

$$X \mapsto \text{Hom}_{\text{cts gp}}(H, \mathcal{O}(X)^\times)$$

is representable by wt space $(\text{Hom}(W, X))$

$$\text{Hom}(H, \mathbb{C}_p^\times)$$

Let's call it W_H . W_H is a finite union of open n -balls

If $H \rightarrow J$ is a cts ep hom

then $W_J \rightarrow W_H$

$$(x^n - 1 = 0)$$

If $H \xrightarrow{\text{injection}} J$ with finite cok. then $W_J \rightarrow W_H$ is an étale surjection of degree $[J:H]$

Examples of H 's

If K is a \mathbb{H} field, \mathcal{O}_K , then $\mathbb{C}_p^\times = (\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$ will do.

More refined example:

if $U \subseteq \prod_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}}^\times$ is opt open.

2. $\Gamma = (\mathcal{O}_K^\times)_{\neq 0} \cap U$, then Γ is a finite index subgroup of \mathcal{O}_K^\times

& $\mathbb{F} / \mathcal{O}_K^\times$ is another example

RR: $\mathbb{F} / \mathcal{O}_K^\times \cong \mathbb{Z}_p^n$ for some n , but n is hard to compute. $\mathbb{Z}^d \subset \mathbb{Z}_p^d$ n should be $[K:\mathbb{Q}] - \text{rk}(\mathcal{O}_K^\times)$

More global example:

$$J = \mathbb{A}_k^x / \mathbb{K}^x \Big/ U(\mathbb{K}_\infty^x)^\circ$$

RR: this is an example because $\mathbb{F} \Big/ \mathbb{C}_P^x \rightarrow J$
& cohen is finite

Define the p -adic eigenvarieties of

tame level U for GL_1/\mathbb{K} to be $\text{Hom}(J, \mathbb{C}_P^x) = \mathcal{L}_U$

Define $W_U = \text{Hom}(H, \mathbb{C}_P^x)$

Get a finite étale map

$$\mathcal{L}_U$$

$$\downarrow$$

$$W_U$$

RR: W_U is a closed subspace of $W_1 = \text{Hom}(O_P^x, \mathbb{C}_P^x)$

& an open subspace of

$$W_2 = \varinjlim_{\substack{\mathbb{F} \subset O_K \\ \text{fin index}}} \text{Hom}\left(\mathbb{A}_k^x / \mathbb{F}^x, \mathbb{C}_P^x\right)$$

What we have done?

We've constructed a p -adic obj.

containing alg. GP's of tame level U .

IF $x \in \mathcal{L}_U$, then $\exists \rho_x$ associated to x .

because x is a map $\mathbb{A}_k^x / \mathbb{K}_\infty^x \Big/ U \rightarrow \mathbb{C}_P^x$
 \uparrow
 $\text{Gal}(\mathbb{F}/\mathbb{K})$

If ρ_x is De Rham then x is an alg. GC

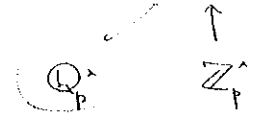
(To know if ρ_x is HT. only depends on $\rho_x|_{D_{\mathbb{G}}}$ s.t.f.)

ie on $x|_{K^x}$ & indeed only

$$\begin{aligned} x|_{\mathbb{Q}_p^x} &= \text{wt of } x \\ &= \text{image of } x \text{ on } W_U \end{aligned}$$

If $K = \mathbb{Q}$, & $U = \prod_{\ell \neq p} \mathbb{Z}_{\ell}^{\wedge}$

then $K^x / A_K^x / U((K_0^x)^{\circ})$ & this is isom.



Chenevra stuff. (BU talk)

$$\text{Hom}(\mathbb{Q}_p^x, \mathbb{Q}_p^x) \cong W \times \mathbb{G}_m$$

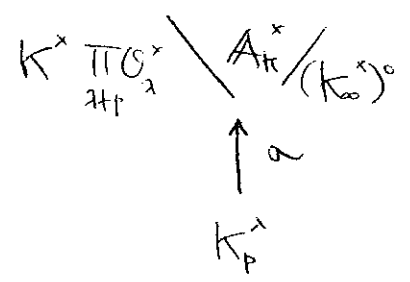
$$\begin{aligned} \mathcal{E}_U &= W = \text{Hom}(\mathbb{Z}_p^x, \mathbb{Q}_p^x) \\ &\downarrow \parallel \\ W_U &= W \end{aligned}$$

if level is small, then map $\mathcal{E}_U \rightarrow W \times \mathbb{G}_m$ is an injection

p -adic ξ -form is a form on \mathcal{E}_U .

One can no doubt check that all p -adic L -forms

live on $\text{Im } \mathcal{E}_U$.



If σ is a surjection then $\mathcal{E}_U = W_U$

Complex world

$\text{Ind}(\chi_1, \chi_2)$

$GL_1 \times GL_1 \subseteq GL_2$

$L(\chi_1, s) \times L(\chi_2, s)$

\mathbb{C}^2

Inductive limit of Tamehot top.