Mar 23. Thu. 2006 2:00 – 3:00pm. Kevin Buzzard

12th lecture.

G : reductive group

$\left\{ \begin{array}{c} \text{Auto. rep'n} \\ \text{for } G \end{array} \right\} \supseteq \left(\begin{array}{c} \text{algebraic} \\ \text{cuspidal} \\ \text{auto. rep'ns} \end{array} \right) \quad \hookrightarrow \quad$ Eigenvarieties ?

Naive

If $G = GL_2/\mathbb{Q}$, cuspidal alg auto. rep'ns

$\longleftrightarrow$ twists of classical cusp new forms $\hookrightarrow$ (Coleman-Mazur Eigen Curve) $\times W$

+ certain Maass forms + (discrete set) $\times W$

Conti. even $G_{\mathbb{Q}} \to GL_2(\mathbb{C})$

Unpublished work of Stevens and others

– gives a family of p-adic L-ftns

on the eigen curve.

Other examples

1) $G = \mathrm{Res}_{K/\mathbb{Q}} GL_1$ K: a number field.

G is an alg group over $\mathbb{Q}$ s.t if A is a $\mathbb{Q}$-alg.

$$G(A) = (A \otimes_{\mathbb{Q}} K)^\times$$

e.g. $G(\mathbb{C}) = (\mathbb{C} \otimes K)^\times \cong \mathbb{C}^{\times \$}$

G is a torus

The automorphic rep's for G biject naturally with

Grössencharacters for $\mathbb{A}_K$

Recall. if $K/\mathbb{Q}$ finite,

$$\mathbb{A}_K = \prod_{v:\text{ a place of } K}' K_v$$

e.g. $\mathbb{A}_{\mathbb{Q}}$ is the subring of $(\prod_p \mathbb{Q}_p) \times \mathbb{R}$

of elements $(x_v)_v$ s.t $x_p \in \mathbb{Z}_p$ for all but fin. many p

$\mathbb{A}_K = \mathbb{A}_{\mathbb{Q}} \otimes_{\mathbb{Q}} K$.

$\mathbb{A}_K^\times$ = idèles

A G.C. is cts gp hom.

$$K^\times \backslash \mathbb{A}_K^\times \longrightarrow \mathbb{C}^\times$$

usual
cx topology.

Facts about $\mathbb{A}_K^\times$

1) If $\mathcal{O} = \mathcal{O}_K$, then $\prod_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}}^\times \subseteq \mathbb{A}_K^\times$ ($\mathfrak{p}$ a fin prime)

& if we define

$$K_\infty = K \otimes_{\mathbb{Q}} \mathbb{R} = \prod_{v|\infty} K_v \cong \mathbb{R}^{n_1} \times \mathbb{C}^{n_2}$$

$$K_\infty^\times \cong (\mathbb{R}^\times)^{n_1} \times (\mathbb{C}^\times)^{n_2} \qquad (K_\infty^\times)^\circ = (\mathbb{R}_{>0})^{n_1} \times (\mathbb{C}^\times)^{n_2} = \text{conn comp. of } 1$$

Then the following is exact,

$$0 \to (\mathcal{O}_K^\times)_{>0} \xrightarrow{\text{diagonally}} K^\times \times \prod_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}}^\times \times (K_\infty^\times)^\circ \longrightarrow \mathbb{A}_K^\times \longrightarrow \text{narrow class gp of } K \longrightarrow 0$$

(f-g) fin gen; diag; finite; infinite; (finite gp)

e.g. $K = \mathbb{Q}$: deduce

$$\mathbb{Q}^\times \times \prod_p \mathbb{Z}_p^\times \times \mathbb{R}_{>0} \cong \mathbb{A}_{\mathbb{Q}}^\times$$

$\therefore$ a GC of $\mathbb{Q}$ is a cts gp homo. $\left(\prod_p \mathbb{Z}_p^\times\right) \times \mathbb{R}_{>0} \longrightarrow \mathbb{C}^\times$

A cts gp hom $\hat{\mathbb{Z}}^\times \longrightarrow \mathbb{C}^\times$ is a Dirichlet char. $\overset{\text{is}}{\hat{\mathbb{Z}}^\times}$

A cts gp hom. $\mathbb{R}_{>0} \longrightarrow \mathbb{C}^\times$ is a complex number s

$$x \longmapsto x^s \quad s \in \mathbb{C}$$

So for $K = \mathbb{Q}$, a GC is a pair (χ, s) $\begin{pmatrix} \chi = \text{a Diri. cha} \\ s \in \mathbb{C} \end{pmatrix}$

Another fact about $\mathbb{A}_K^\times$.

$$\mathrm{Gal}(\overline{K}/K)^{ab} = \underbrace{\overline{K^\times \backslash \mathbb{A}_K^\times / (K_\infty^\times)^\circ}}_{}$$

(G. CFT)

$$K^\times \backslash \mathbb{A}_K^\times \sim \frac{\hat{\mathcal{O}}^\times \cdot K_\infty^\times}{\mathcal{O}^\times}$$

$$\mathbb{A}_K^\times \Big/ \overline{K^\times \cdot (K_\infty^\times)^\circ}$$

Say $\chi : K^\times \backslash \mathbb{A}_K^\times \longrightarrow \mathbb{C}^\times$ is a GC.

$$\chi_\infty := \chi|_{(K_\infty^\times)^\circ} : (\mathbb{R}_{>0})^{n_1} \times (\mathbb{C}^\times)^{n_2} \longrightarrow \mathbb{C}^\times$$

Say χ is <u>algebraic</u>

If χ_∞ is of the form

$$(x_1, \dots, x_{n_1}, z_1, \dots, z_{n_2}) \longmapsto \prod_i x_i^{m_i} \times \prod z_j^{r_j} \bar{z}_j^{s_j}$$

with m_i & r_j & $s_j \in \mathbb{Z}$

(Gro Hec char) $\supseteq$ (alg GCs)

Tate observed that the set of all GCs naturally formed a complex analytic manifold.

K: $\mathbb{Q}$: ∞ly many copies of $\mathbb{C}$

Any K ——

Reason is $\mathbb{A}_K^\times \longrightarrow \mathbb{R}_{>0}$

$(a) \longmapsto Na = \prod_v |a_v|$ $|2|=4$

Product formula $\Rightarrow K^\times \subseteq$ kernel.

Get a GC $\|\cdot\|$: $K^\times \backslash \mathbb{A}_K^\times \longrightarrow \mathbb{C}^\times$ ⟶ cyclotomic character

Put 2 GCs χ_1 & χ_2 into same cpt

$\Longleftrightarrow$ $\chi_1 / \chi_2 = \|\cdot\|^s$, some $s \in \mathbb{C}$.

<u>Tate</u>: If χ is a GC

he defined $L(\chi) \in \mathbb{C} \cup \{\infty\}$ s.t resulting fctn on R.S. (Riemann Surface) is meromorphic

If χ is an <u>alg</u> GC, then $\exists$ compatible system of λ-adic Galois repns associated to χ.

Idea of construction

$\chi_\infty = \chi|_{(K_\infty^\times)^\circ}$. Extend χ_∞ to all of $K_\infty^\times$ in obvious way (same formula)

$\chi \div \chi_\infty$: $\mathbb{A}_K^\times \longrightarrow \mathbb{C}^\times$ trivial on $(K_\infty^\times)^\circ$. (non-trivial on $K^\times$)

Check: image of $\chi \div \chi_\infty$ is contained in a number field E.

If λ is a prime of E, $\lambda | \ell$, then $\cancel{(\chi \div \chi_\infty)}|_{K_\ell^\times = (K \otimes \mathbb{Q}_\ell)^\times}$

then $(\chi \div \chi_\infty)\big|_{K^\times} : K^\times \longrightarrow E^\times$

$K^\times$ embedded diagonally

can be extended ctsly to a map

to a map $\psi : K_\ell^\times \longrightarrow E_\lambda^\times$

$K_\ell^\times = (K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^\times$

$\chi \div \chi_\infty \div \psi$ is now trivial on $K^\times$ & $(K_\infty^\times)^\circ$

induces a map.

$\chi_\lambda : Gal(\bar{K}/K)^{ab} \longrightarrow E_\lambda^\times$.

The χ_λ are all Hodge-Tate & unramified outside a finite set.

Rk: If $\alpha : Gal(\bar{K}/K) \longrightarrow GL_1(E_\lambda)$

is H-T. & unram. outside a finite set

then arises from a GC in this way.

& Hence in particular compatible family

(Alg. GCs) $\xhookrightarrow{\text{Game}}$ (p-adic Theory)

Fix embeddings $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$

$K^\times \longrightarrow \mathbb{C}^\times \quad x \mapsto \prod_\sigma \sigma(x)^{-n_\sigma}$

$\chi(\underline{a}) \div \chi_\infty(\underline{a}_\infty) - \psi(a_\ell)$

Now given χ, an alg. GC,

get $\chi_p : \mathbb{A}_K^\times / \overline{K^\times \cdot (K_\infty^\times)^\circ} \longrightarrow \mathbb{C}_p^\times$

$\ker \chi \supseteq \mathcal{O}_\wp^\times$ for all but fin. many $\wp$

but not all conti maps.

$\mathbb{A}_K^\times / K^\times \cdot (K_\infty^\times)^\circ \longrightarrow \mathbb{C}_p^\times$ arise in this way

$\chi(\mathcal{O}_\wp) \subset \mathcal{O}_{\mathbb{C}_p}^\times$

Fix now a tame level

$U \subseteq \prod_{\substack{\lambda \text{ fin} \\ \lambda \nmid p}} \mathcal{O}_\lambda^\times$ <u>cpt open</u>

e.g. $K = \mathbb{Q}$

if $N \geq 1$, $p \nmid N$

$U_N = \{ \alpha \in \prod_{\ell \neq p} \mathbb{Z}_\ell^\times, \ \alpha \equiv 1 \bmod N \}$

General lemma

If H is an abelian profinite gp. s.t H contains an open subgp iso. to $(\mathbb{Z}_p)^n$ for some $n \geq 0$.
then $\mathrm{Hom}_{\substack{cts\\gp}}(H, \mathbb{C}_p^\times)$ is naturally a $\mathbb{C}_p$-points of rigid space over $\mathbb{C}_p$

Functor: rigid spaces $\longrightarrow$ gps.

$X \longmapsto \mathrm{Hom}_{\substack{cts\\gp}}(H, \mathcal{O}(X)^\times)$

is representable $(\mathrm{Hom}(W, X))$ by wt space

e.g $H = \mathbb{Z}_p$: space is

$\{w \in \mathbb{C}_p : |w-1| < 1\}$

$\cong$ open ball B

If $H = \mathbb{Z}_p^n$. space is B^n. $H =$ fin. gp. space is discrete. $\mathrm{Hom}(H, \mathbb{C}_p^\times)$

Let's call it W_H. W_H is a finite union of open n-balls

If $H \to J$ is a cts gp hom

then $W_J \longrightarrow W_H$

$(X^n - 1 = 0)$

If $H \underset{\text{injection}}{\hookrightarrow} J$ with finite coker then $W_J \to W_H$ is an "étale" surjection of degree $[J:H]$

Examples of H's.

If K is a # field, $\mathcal{O}_K$. then $\mathcal{O}_p^\times = (\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$ will do.
$\quad \| \quad \mathcal{O}_{K_p}^\times$

More refined example:

if $U \subseteq \prod_{\lambda \nmid p} \mathcal{O}_\lambda^\wedge$ is cpt open.

& $\Gamma = (\mathcal{O}_K^\times)_{>0} \cap U$, then Γ is a finite index subgp of $\mathcal{O}_K^\times$

& $\overline{\Gamma} \backslash \mathcal{O}_{K_p}^\times$ is another example
$\,\!\!\|\!\!\,$ H

Rk: $\overline{\Gamma} \backslash \mathcal{O}_p^\times \overset{\text{open}}{\supseteq} \mathbb{Z}_p^n$ for some n, but n is hard to compute

$\mathbb{Z}^e \subset \mathbb{Z}_p^d$ $\quad$ n <u>should</u> be $[K:\mathbb{Q}] - \mathrm{rk}(\mathcal{O}_K^\times)$

More global example:

$$J = {}_{K^\times}\backslash \mathbb{A}_K^\times / U(K_\infty^\times)^\circ$$

<u>Rk</u>: this is an example because $\overline{\Gamma}\backslash \mathbb{C}_p^\times \longrightarrow J$ & coker is finite as it's class gp.

Define the p-adic eigenvarieties of tame level U for GL_1/K to be $\mathrm{Hom}(J, \mathbb{C}_p^\times) = \mathcal{E}_U$

Define $\mathcal{W}_U = \mathrm{Hom}(\underset{\overline{\Gamma}\backslash \mathcal{O}_p^\times}{\overset{\shortparallel}{H}}, \mathbb{C}_p^\times)$

Get a finite étale map

$$\begin{array}{c} \mathcal{E}_U \\ \downarrow \\ \mathcal{W}_U \end{array}$$

[<u>Rk</u>: $\mathcal{W}_U$ is a closed subspace of $\mathcal{W}_1 = \mathrm{Hom}(\mathcal{O}_p^\times, \mathbb{C}_p^\times)$

& an open subspace of

$$\mathcal{W}_2 = \varinjlim_{\substack{\Gamma \subset \mathcal{O}_K^\times \\ \text{fin index}}} \mathrm{Hom}\left({}_{\overline{\Gamma}}\backslash \mathcal{O}_p^\times, \mathbb{C}_p^\times\right)$$

<u>What we have done?</u>

We've constructed a p-adic obj. containing alg. GCs of tame level U.

If <u>$x \in \mathcal{E}_U$</u>, then $\exists\, \rho_x$ associated to x

because x is <u>a map</u> ${}_{K^\times}\backslash \mathbb{A}_K^\times / K_\infty^\times \cdot U \longrightarrow \mathbb{C}_p^\times$

$$\begin{array}{c} \uparrow \\ \mathrm{Gal}(\overline{K}/K) \end{array}$$

If ρ_x is De Rham then x is an alg. GC

(To know if ρ_x is HT only depends on $\rho_x|_{D_\wp}$, $\wp | p$.)

i.e on $x|_{K_p^\times}$ & indeed only

$x|_{\mathcal{O}_p^\times}$ = wt of x = image of x in $\mathcal{W}_U$

If $K = \mathbb{Q}$, & $U = \prod_{\ell \neq p} \mathbb{Z}_\ell^\times$

then $K^\times \backslash \mathbb{A}_K^\times / U (K_\infty^\times)^\circ$ & this is bom.

$$\begin{array}{ccc} & K^\times \backslash \mathbb{A}_K^\times / U (K_\infty^\times)^\circ & \\ & \uparrow & \\ \mathbb{Q}_p^\times & \mathbb{Z}_p^\times & \end{array}$$

Chenevier stuff (BU talk)

$\mathrm{Hom}(\mathbb{Q}_p^\times, \mathbb{C}_p^\times) \cong \mathcal{W} \times \mathbb{G}_m$

$$\begin{array}{ccc} \mathcal{E}_U = \mathcal{W} & = & \mathrm{Hom}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times) \\ \downarrow & & \| \\ \mathcal{W}_U & = & \mathcal{W} \end{array}$$

if level is small, then map $\mathcal{E}_U \to \mathcal{W} \times \mathbb{G}_m$ is an injection

p-adic ζ-fn is a fn on $\mathcal{E}_U$.

One can no doubt check that all p-adic L-fns live on $\varinjlim_U \mathcal{E}_U$.

$$\begin{array}{c} K^\times \prod_{\lambda \nmid p} \mathcal{O}_\lambda^\times \backslash \mathbb{A}_K^\times / (K_\infty^\times)^\circ \\ \uparrow \alpha \\ K_p^\times \end{array}$$

If α is a surjection then $\mathcal{E}_U = \mathcal{W}_U$

Complex world

$\mathbb{C}^2$

$\mathbf{Ind}(\chi_1, \chi_2)$

$GL_1 \times GL_1 \subseteq GL_2$

$\mathbf{L}(\chi_1, s) \times \mathbf{L}(\chi_2, s)$

$\mathbb{C}^2$

Inductive limit of Frechet top.