Feb 14 Tuesday. 2:00pm – 2:30pm. Kevin Buzzard 4th ~~talk~~ lecture (E.V)

21 & 23rd Feb (No lecture)

Examples of cpt ops with CPS (char power series) $= 1$. $I = \mathbb{N}$

i) $\varphi = 0$.

ii) $\begin{pmatrix} 0 & 1 & & & \\ & 0 & \pi & & \\ & & 0 & \pi^2 & \\ & & & 0 & \ddots \end{pmatrix}$ kernel $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$ $\pi \in K$, $0 < |\pi| < 1$

iii) $\begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & \pi & 0 & \\ & & \pi^2 & \ddots \end{pmatrix}$ image is not dense

iv) Define $\sigma: \mathbb{N} \to \mathbb{N}$

$$\begin{array}{ccccccc} 1 & \xleftarrow{\sigma} & 3 & \xleftarrow{\sigma} & 5 & \xleftarrow{\sigma} & 7 \\ \downarrow & & & & & & \\ 2 & \longrightarrow & 4 & \longrightarrow & 6 & \longrightarrow & 8 \end{array}$$

& define φ by $\varphi(e_i) = \pi^i e_{\sigma(i)}$

φ is cpt, inj, dense image & cps $= 1$

Last few facts about cpt op

i) cpt $\circ$ cts $=$ cpt
& cts $\circ$ cpt $=$ cpt [pf: true for finite rk operators]

ii) If $a: V \to W$ is cts
& $b: W \to V$ is cpt.
then $a \circ b$ & $b \circ a$ are cpt & $\mathrm{cps}(a \circ b) = \mathrm{cps}(b \circ a)$

Recall $\mathrm{cps}(\varphi)$ is a power series in X that converges $\forall X \in K$

Crash course in $K\langle T \rangle$

K is still a field complete w.r.t non-trivial non-arch valuation

$K\langle T \rangle$ is the ring $\{ \sum_{n \geq 0} a_n T^n \ ; \ a_n \in K, \ a_n \to 0 \text{ as } n \to \infty \}$

If $\mathcal{O}$ is $\{x \in K : |x| \leq 1\}$
& if $\pi \in \mathcal{O}$ s.t $0 < |\pi| < 1$, then $K\langle T \rangle = K \otimes_{\mathcal{O}} \varprojlim (\mathcal{O}/\pi^n \mathcal{O})[T]$

Remark : $K\langle T\rangle$ is precisely the power series $\sum a_n T^n \in K[[T]]$ which formally converge when you evaluate at $T=t$, $\forall t\in \mathcal{O}$.

Two natural "norms" on $K\langle T\rangle$: first is $|\sum a_n T^n| = \max_{n\geq 0} |a_n|$

2nd : If L is any finite ext'n of K then there's a unique extension of norm of K to L

& one can compute $\sup_{\substack{L/K \text{ finite} \\ t\in \mathcal{O}_L}} |\sum a_n t^n|_L$ ⇝ this is bdd by $\max_{n\geq 0}|a_n|$, so it exists

In fact this sup is $\max|a_n|$

[pf. WLOG $\max|a_n|=1$

& we need to find L & $t\in\mathcal{O}_L$ s.t. $|\sum a_n t^n|_L = 1$

If m = max ideal of $\mathcal{O}$ & $f=\sum a_n T^n$ with $\max|a_n|=1$

then $\bar{f} = \sum \bar{a}_n T^n \in k[T]$ is non-zero

$x^p - x$ / $K=\mathbb{Q}_p$

$k = \mathcal{O}/m$ $\bar{a}$ = red. of $a\in\mathcal{O}$ to k

$\sup_{t\in K} \cancel{|\sum a_n t^n|_K}$

$\exists\, \tau \in$ fin. ext'n l of k s.t. $\bar{f}(\tau)\neq 0$.

Lift l to L lift τ to t.

Another nice fact "The Tate alg"

If $f, g \in K\langle T\rangle$, then $|fg| = |f|\cdot|g|$

Pf WLOG $|f|=|g|=1$ Then $\bar{f}, \bar{g}\neq 0$.

$\therefore \bar{f}\,\bar{g}\neq 0$ as $k[T]$ is an ID

$\therefore |fg| = 1$

If $0\neq f\in K\langle T\rangle$, $f=\sum a_n T^n$ then there's a unique $s\in\mathbb{Z}_{\geq 0}$ s.t. $|f|=|a_s|$ & $|a_s|>|a_t|$ $\forall t>s$

We say f is s-distinguished ← special case. (if $|f|=1$ then $s=\deg(\bar{f})$)

Weierstraß preparation Weierstraß division thm.

If f is s-distinguished, then $\forall g\in K\langle T\rangle$ $\exists!$ $q, r\in K\langle T\rangle$ with $g = q\cdot f + r$ & $r\in K[T]$ $\deg(r)<s$ & furthermore $|g| = \max\{|q|\cdot|f|, |r|\}$

Proof) [BGR] p. 200

Weierstraß Preparation

If f is s-distinguished then $\exists!$ monic poly(m) of deg. s

& $u \in K\langle T\rangle^{\times}$ s.t. $f = m \cdot u$

pf) Set $g = X^s$. $X^s = g \cdot f + r$ (f s-distinguished). Set $m = X^s - r = g \cdot f$

$u^{-1} = g$

$f = 1 + pX$

(0)-disting

$m = 1$.

$|X^s| = 1 = \max\{|g||f|, |r|\}$

$\begin{cases} \deg(r) < s. \\ |r| \leq 1 \end{cases}$

WLOG $|f| = 1$

$|g| \leq 1$

$\bar{X}^s = \bar{g}\bar{f} + \bar{r}$, $\bar{f}$ deg s

$\deg(\bar{g}) = 0$

($\boxed{\bar{g} \in k^*}$

$g \in K\langle T\rangle^{\times}$

One last remark.

$K\langle T\rangle$ is a Banach sp over K & naturally ON-able with a basis $1, T, T^2, T^3$.

If $\pi \in K$ $|\pi| < 1$.

There's a map $K\langle T\rangle \to K\langle S\rangle$

$T \mapsto \pi S$ — a ring hom.

$\sum a_n T^n \mapsto \sum a_n \pi^n S^n$.

& w.r.t the obvious ON basis

this has matrix $\begin{pmatrix} 1 & & & \\ & \pi & & \\ & & \pi^2 & \\ & & & \pi^3 & \\ & & & & \ddots \end{pmatrix}$ it's cpt.

Slopes

Recall if $\varphi : V \to V$ is cpt. then it has a char power series $CPS(\varphi) = \sum a_n T^n$ $\lambda^n |a_n| \to 0$. $\forall \lambda \in \mathbb{R}_{>0}$.

If $a \in K$ was a zero of $CPS(\varphi)$ of order h

then $V = N \oplus F$ with $\dim N = h$

& $1 - a\varphi$ nil on N. invertible on F.

N is morally the generalized eigenspace corresponding to the eigen value $\frac{1}{a}$

Idea of slopes: if $F = \sum a_n T^n$

& $\lambda^n |a_n| \to 0$. $\forall \lambda \in \mathbb{R}_{>0}$

(i.e. F converges on all of K)

the Newton polygon, of F, which tells you a lot about the zeroes of F.

Assume $\underline{a_0 = 1}$ $\qquad F = 1 + a_1 T + \cdots$

We have $|\cdot| : K \longrightarrow \mathbb{R}_{\geq 0}$

$|x| = 0 \iff x = 0$

Now choose your favorite positive real # c

& define $v : K^\times \longrightarrow \mathbb{R}$ by $v(x) = -c \log |x|$

e.g. $K = \mathbb{Q}_p$

$|p| = p^{-1}$

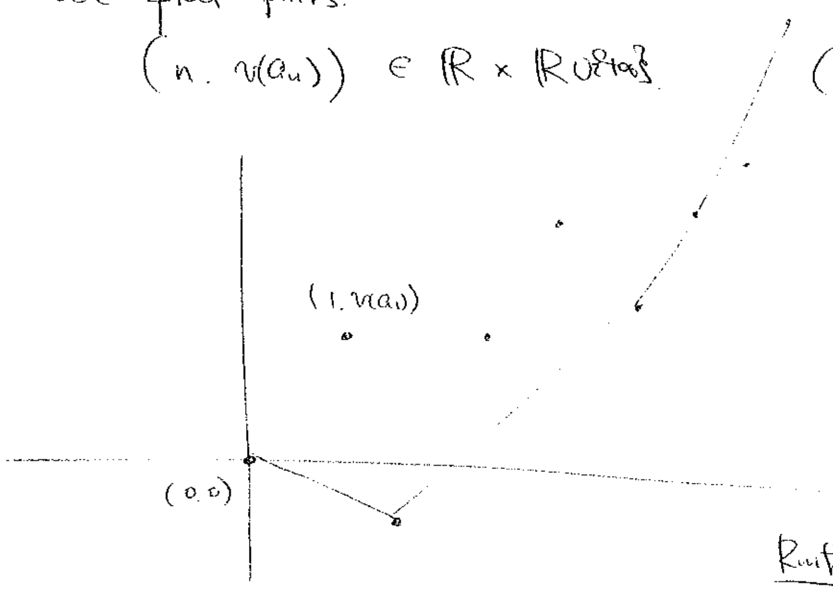
$c = \frac{1}{\log p}$. $v(p) = 1$

Formally extend v to K by $v(0) = +\infty$

Now for $F = 1 + \cdots = \sum a_n T^n$.

We plot pairs.

$(n, v(a_n)) \in \mathbb{R} \times \mathbb{R} \cup \{+\infty\}$ $\qquad (n \geq 0)$



Now take "lower convex hull" of this diagram

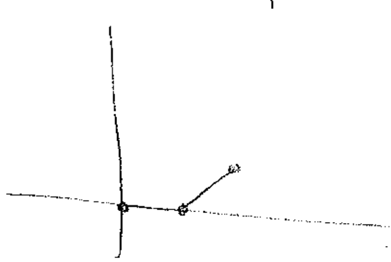
Rmk. F converges on all of K

$\Rightarrow \forall \varepsilon, \exists N, \ |a_n| \leq \varepsilon^n \ \forall n \geq N$

$\Rightarrow \left(\begin{array}{l} \forall K, \exists N \text{ s.t. } \\ v(a_n) \geq K \cdot n \\ \text{for all } n \geq N \end{array} \right.$

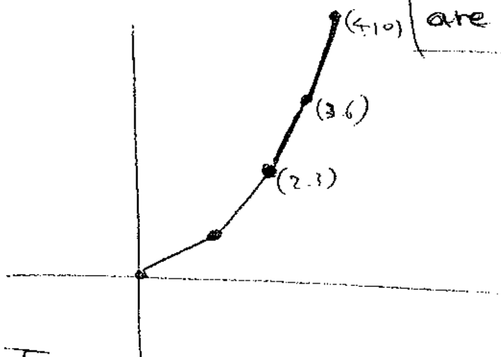
Example. $K = \mathbb{Q}_p$

$F(X) = 1 + X + pX^2$



$f(x) = \prod_{n \geq 1} (1 - p^n X)$

Coeff of X^i has valuation $\frac{i(i+1)}{2}$ as its sum of only many terms
one of which is $\pm p^{\frac{i(i+1)}{2}}$ & the rest of which are all smaller.



Newton poly is infinitely many lines, the mth of which has slope m.

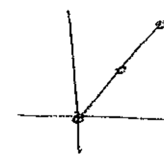
For a general Newton polygon, take a line that's a "part" of it (i.e. ends of line @ consecutive vertices of Np).

Slope of line is $\frac{\text{opp.}}{\text{adj}}$

Length = length of projection onto X-axis

Basic facts: slopes of NP

$1 + pX + p^2X^2$



are precisely valuations of reciprocals of zeros of power series (in an alg. closure of K). & multiplicities are determined by lengths.

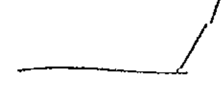
e.g.) $1 + pX + p^2X^2$ has 2 roots, each with valuation -1

$1 + X + pX^2$ has 2 roots one with $v = 0$, one with $v = -1$

$\prod_{n \geq 1} (1 - p^n X)$ has ∞ly many roots one with val. $-n$ for all $n \geq 1$

Furthermore if $F = \sum a_n T^n \in K\langle T \rangle$ & $K \subseteq L$ is some huge ext'n of K & if $x \in L$ is a zero of F,

then $\deg [K(x) : K] < \infty$.



Sketch of how to prove this.

"Scale F" $= \sum a_n T^n$ $(T \longrightarrow \lambda T)$ until $a_n \in \mathcal{O}_K$ for all $n \geq 0$
& F is S-distinguished for some $s \geq 0$

By Weierstraß prep.

$F_1 = m \cdot u$ m a poly $u \in K\langle T\rangle^\times$.

Easy check $|m| = |u| = 1$ & messing around with power series $\Rightarrow$ u converges everywhere

& NP of u = NP of F_1 minus slope 0 part.

Let me finish with one example of when one can actually compute all the slopes of $CPS(\varphi)$

Set $I = \{0, 1, 2, \dots\}$

V corresponding Banach ON basis $e_0, e_1, e_2, \dots$ op.

$\varphi : V \to V$ compact

& (a_{ji}) = matrix of φ.

Assume $\exists$ constants $d_0, d_1, d_2, \dots \in K$

s.t. $|d_0| > |d_1| > |d_2| > \dots > |d_n| > \dots$

& $|d_i| \to 0$ as $i \to \infty$

s.t. $|a_{ij}| \le |d_i|$ $\forall i, \forall j$

Pictorially i^{th} row is a multiple of d_i in $\mathcal{O}$.

$$\begin{array}{c} d_0\mathcal{O} \\ d_1\mathcal{O} \\ d_2\mathcal{O} \end{array} \begin{pmatrix} u \cdot d_0 & \cdots & & \\ & \cdot & \cdot & \\ & & \cdot\ \cdot & \end{pmatrix}$$

& that if $b_{ij} = \dfrac{a_{ij}}{d_i}$

then $\det(b_{ij})_{0 \le i, j \le n-1} \in \mathcal{O}^\times$ for all $n \ge 0$.

Then **NP** of $CPS(\varphi)$ coincides with NP of $\prod_{i \ge 0} (1 - d_i X)$

& In particular, the zeros of $CPS(\varphi)$ are $x_0, x_1, x_2, \dots \in K$

& if they are ordered so $|x_0| > |x_1| > |x_2| > \dots$

then $|x_i| = |d_i|$ $\forall i \ge 0$

Pf. elementary

Recall if $S \subseteq I$ was finite & $\sigma : S \xrightarrow{\sim} S$

then $n_{\sigma,S} = \prod_{i \in S} a_{i\,\sigma i}$

$$n_S = \sum_{\sigma : S \to S} \mathrm{sgn}(\sigma)\, n_{\sigma S} \qquad c_m = (-1)^m \sum_{\#S = m} n_S$$

& $cps(\varphi) = \sum c_m T^m$.

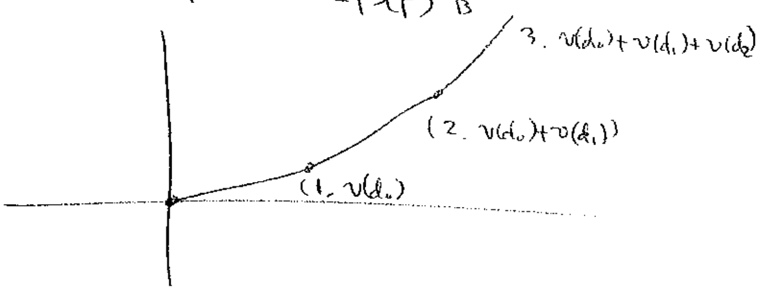
Conditions say

$$v(n_{\sigma,S}) \geq \sum_{s \in S} v(d_s)$$

$\therefore\ v(n_S) \geq \sum_{s \in S} v(d_s) > \sum_{s=0}^{m-1} v(d_s)$ if $S \neq \{0, \dots, m-1\}$

det $\in$ unit $\Rightarrow v(n_{\{0,1,\dots,m-1\}}) = \sum_{s=0}^{m-1} v(d_s) \Rightarrow v(c_m) = \sum_{s=0}^{m-1} v(d_s)$

Hence NP of $cps(\varphi)$ is



& In particular n-th segment has length 1 & slope $v(d_{n-1})$

Next time: Use this to compute slopes of some **modular** forms