Feb 2. Thursday 1:00–2:30pm **Kevin Buzzard.** (255y) 1st lecture

A concrete introduction to p-adic modular forms.

History:

$GL_2^+(\mathbb{R}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})$ det > 0

$GL_2^+(\mathbb{R})$ acts on $\mathfrak{h} := \{x+iy \in \mathbb{C} : y>0\}$ via $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\tau = \frac{a\tau+b}{c\tau+d}$

If Γ is a certain kind of discrete subgp of $GL_2^+(\mathbb{R})$, then we may want to consider holomorphic functions $f : \mathfrak{h} \to \mathbb{C}$ which "transform well under Γ"

e.g. could ask $f(\gamma\tau) = f(\tau)$ $\forall \gamma \in \Gamma$

or more generally $f(\gamma\tau) = j(\gamma,\tau)^k f(\tau)$

where $j\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau\right) := c\tau + d$

$f(\gamma\tau) = f(\tau)$ $\forall \gamma \in \Gamma$

is the same as saying that f is a fctn on the quotient $\Gamma\backslash\mathfrak{h} =: Y(\Gamma)$ ← Riemann Surface

Warning: the complex structure on $Y(\Gamma)$ is induced from that of $\mathfrak{h}$.

If $y \in Y(\Gamma)$ is a "general" pt. then one can find a small nbd $y \in U \subseteq Y(\Gamma)$ & $V \subseteq \mathfrak{h}$ open s.t.

$$\begin{array}{ccc} V & \subseteq & \mathfrak{h} \\ \downarrow\wr & & \downarrow \\ U & \subseteq & Y(\Gamma) \end{array}$$

If $\exists \gamma \in \Gamma$ & $\tau \in \mathfrak{h}$ s.t. $\gamma \neq$ scalar but $\gamma\tau = \tau$

then, in our case γ will always have finite order (cyclic)

& $\exists$ diagram of the form

$$\begin{array}{ccccccc} z & & D & \cong & V & \subseteq & \mathfrak{h} \\ \downarrow & & \downarrow & & \downarrow & & \\ z^e & & D & \cong & U & \subset & Y(\Gamma) \end{array}$$

(D = open unit disk) for some $e \geq 1$

$\Gamma\backslash\mathfrak{h}$

[Minor miracle: quotient of open unit disc by a finite gp **of rotations** is again open unit disc]

$f : \mathfrak{h} \to \mathbb{C}$ s.t. $f(\gamma\tau) = f(\tau)$ $\forall \gamma \in \Gamma$

↕ holo. fctn on $Y(\Gamma)$

What about the $(c\tau+d)^k$ factor?

If Γ has **no** elliptic points, then holo fns $f : \mathfrak{h} \to \mathbb{C}$ s t $f(\gamma\tau) = (j(\gamma,\tau))^k f(\tau)$ are precisely holomorphic sections of a holomorphic line bundle $\omega^{\otimes k}$ on $Y(\Gamma)$.

$$\begin{array}{c} \mathfrak{h} \times \mathbb{C} \\ \downarrow \\ \mathfrak{h} \end{array}$$

Define an action of Γ on $\mathfrak{h} \times \mathbb{C}$

via $\gamma(\tau, z) = (\gamma\tau, \; j(\gamma,\tau) \cdot z)$

If Γ has no elliptic pts. then $\Gamma \backslash \mathfrak{h} \times \mathbb{C}$ is a holomorphic line bundle

$$\begin{array}{c} \Gamma \backslash \mathfrak{h} \times \mathbb{C} \\ \downarrow \\ \Gamma \backslash \mathfrak{h} = Y(\Gamma) \end{array}$$

called $\boldsymbol{\omega}$ on $Y(\Gamma)$

If no elliptic pts $f : \mathfrak{h} \to \mathbb{C}$

$f(\gamma\tau) = j(\gamma,\tau)^k f(\tau)$ = sections of $\omega^{\otimes k}$

Trick for understanding the equation $f(\gamma\tau) = (c\tau + d)^k \cdot f(\tau)$ geometrically when Γ has elliptic points

in all the cases we're interested in, Γ will have a normal subgp Δ of finite index s.t Δ has no elliptic pts. elements

ω exists on $Y(\Delta)$

& we can consider holomorphic sections of $\omega^{\otimes k}$ on $Y(\Delta)$

the gp Γ/Δ acts naturally on these sections

from analytic pt of view; if $\gamma \in GL_2^+(\mathbb{R})$ & $f\ \mathfrak{h} \to \mathbb{C}$

define $f|_k\gamma : \mathfrak{h} \to \mathbb{C}$ by $(f|_k\gamma)(\tau) = j(\gamma,\tau)^{-k} \cdot f(\gamma\tau)$

Eqn simplifies to $f|_k\gamma = f \quad \forall \gamma \in \Gamma$

If $f|_k \delta = f \quad \forall \delta \in \Delta$ & $\gamma \in \Gamma$, then $f|_k\gamma$ is now invariant under $\gamma^{-1}\Delta\gamma$ $= \Delta$

Geometric definition for Γ with elliptic pts

Choose Δ as above

& consider Γ/Δ - invariant sections of $\omega^{\otimes k}$ on $Y(\Delta)$

The Γ's I am interested in are gps: $SL_2(\mathbb{Z})$
& its finite index subgps $\Gamma_0(N)$, $\Gamma_1(N)$ & $\Gamma(N)$; if $N \in \mathbb{Z}_{\geq 1}$
then these are subgps of $SL_2(\mathbb{Z})$
defined by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ s.t.

$c \equiv 0 \ (N)$	$\Gamma_0(N)$
$c \equiv 0$ & $a \equiv d \equiv 1 \ (N)$	$\Gamma_1(N)$
$c \equiv b \equiv 0$ & $a \equiv d \equiv 1 \ (N)$	$\Gamma(N)$

Unfortunately,
there are too many $f : \mathfrak{h} \to \mathbb{C}$ s.t. $f|_k \gamma = f \quad \forall \gamma \in \Gamma$

essentially because $Y(\Gamma)$ is never cpt for these Γ if Γ is $\Gamma_0(N)$ etc as above

e.g. $Y(SL_2(\mathbb{Z})) \underset{j}{\cong} \mathbb{C}$

Standard theory of compactification: if $\Gamma \subseteq SL_2(\mathbb{Z})$ is as above, then one adds $\Gamma \backslash \mathbb{Q} \cup \{\infty\}$ to $Y(\Gamma)$ & the resulting set $X(\Gamma)$ is naturally a cpt connected Riemann surface
if no elliptic pts.

We have ω on $Y(\Gamma)$ $\wedge$ Does ω extend naturally to $X(\Gamma)$?

Not in general; problems at "irregular cusps"

$$\text{e.g. } \Gamma = \Gamma_1(4)$$

no elliptic pts
1 - irregular

<u>Fact</u>: $\Gamma_1(N)$ has no elliptic pts & no irregular cusps.
if $N \geq 5$.

$X_1(N) = X(\Gamma_1(N))$ has a sheaf ω

<u>Def'n</u>:

	Compactifications	
$Y_0(N) = Y(\Gamma_0(N))$	$X_0(N)$	All cpt connected
$Y_1(N) = Y(\Gamma_1(N))$	$X_1(N)$	
$Y(N) = Y(\Gamma(N))$	$X(N)$	Riemann Surfaces

Analytic definition of a modular form of wt k & level $\Gamma = \Gamma_0(N)$, $\Gamma_1(N)$, $\Gamma(N)$

is a holomorphic fn $f : \mathfrak{h} \to \mathbb{C}$ s.t.

$$f(\gamma \tau) = (c\tau + d)^k f(\tau) \quad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

[illegible]

A cusp form is same as above, but with a stronger bddness condition

On geometric side, these defs are much nicer:

<u>If</u> Γ has no elliptic pts & no irregular cusps

then a weight k modular form is a holomorphic section of $\omega^{\otimes k}$ on $X(\Gamma)$

& a cusp form is a holo. section of $\omega^{\otimes k}$ on $X(\Gamma)$ that vanishes on the finite set of cusps $X(\Gamma)\setminus Y(\Gamma)$

$(\text{cusps})+\Omega^1 = \omega^{\otimes 2}$

<u>Notation</u>: if $\Gamma = \Gamma_0(N), \Gamma_1(N), \Gamma(N)$ then $M_k(\Gamma)$ = modular forms of wt k & level Γ

$\cup\!\mathsf{I}$

$S_k(\Gamma)$ = cusp forms of wt k level Γ

<u>Remark</u>:

$M_k(\Gamma)$ (& hence $S_k(\Gamma)$) are <u>finite-dim.</u> (coherent sheaf / coh. sheaf under proper base change)

Say $f \in M_k(\Gamma_1(N))$. Then $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix} \in \Gamma_1(N)$

$$\therefore\ f|_k\gamma = f \qquad \text{i.e. } f(\tau+1) = f(\tau)$$

$$\mathfrak{h}\Big/\tau\longmapsto\tau+1 \ \cong\ \{ q \in \mathbb{C} ;\ 0<|q|<1\} \quad \text{punctured open unit disk}$$

$$\tau \longmapsto e^{2\pi i\tau} =: q$$

$\therefore$ f has a power series expansion $\displaystyle\sum_{-\infty<n<\infty} a_n q^n$. q as above

f extends to $\omega^{\otimes k}$ on $X(\Gamma)$ $\Rightarrow a_n = 0\ \forall n<0$

$f \in S_k(\Gamma) \Rightarrow a_0 = 0$ too

<u>Frequently</u> $f \in M_k(\Gamma)$ is "understood" via <u>this power series expansion</u>

<u>Fact</u>: $M_k(\Gamma_1(N))$ is a computable space, in the sense

for a **given** k & N, a computer program & $M \geq 0$ can efficiently compute the power series expansion up to $O(q^M)$ for a basis of $M_k(\Gamma_1(N))$

Examples of "far less" computable objects:

If $\Gamma \leq GL_2^+(\mathbb{R})$ is a general "arithmetic" subgp, then $M_k(\Gamma)$ may be difficult to compute.

$D \otimes \mathbb{R} \cong M_2(\mathbb{R})$
indefinite quat. alg
$D \supseteq \mathcal{O}_D \supset \mathcal{O}_D^{N=1}$

Another Example:

one could consider C^∞ ftns $f : \mathfrak{h} \longrightarrow \mathbb{C}$ satisfying a similar, but different, differential equation

s.t. $f(\gamma\tau) = f(\tau)$ $\forall \gamma \in \Gamma$ & bdd condition.
$\Gamma(N)$

– these are Maass forms

& it's very hard to compute these too.

Example: Level 1 modular forms

Let $k \geq 4$ be an even integer.

Consider the ftn $\displaystyle\sum_{\substack{m,n \in \mathbb{Z} \\ \text{not both } 0}} \frac{1}{(m\tau + n)^k}$

this is a holomorphic ftn on $\mathfrak{h}$.

This ftn is invariant under $|_k \gamma$ for $\gamma \in SL_2(\mathbb{Z})$

& one can compute the power series expansion of this ftn.

$$2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{n \geq 1} \sigma_{k-1}(n) \cdot q^n \quad \text{where } \sigma_t(n) = \sum_{0 < d \mid n} d^t$$

Dividing by $2\zeta(k)$ we get a ftn

$$E_k(\tau) = 1 + \frac{2}{\zeta(1-k)} \sum_{n \geq 1} \sigma_{k-1}(n) \cdot q^n.$$

[Rmk: this power series is in $\mathbb{Q}[[q]]$ rather than $\mathbb{C}[[q]]$]

[Recall $\zeta(1-k) \in \mathbb{Q}^\times$, if $k \geq 4$ is even]

Let's also define $G_k(\tau) = \dfrac{\zeta(1-k)}{2} \times E_k(\tau)$

$$= \frac{\zeta(1-k)}{2} + \sum_{n \geq 1} \sigma_{k-1}(n) \cdot q^n$$

$\sigma_t(n) = \sum_{0 < d \mid n} d^t$

Hence $\sigma_t(-n) = \sigma_t(n)$

$$= \frac{\zeta(1-k)}{2} + \frac{1}{2} \sum_{n \neq 0} \sigma_{k-1}(n) \cdot q^{|n|}$$

$\sigma_{k-1}(0) = \sum_{d \geq 1} d^{k-1} = \zeta(1-k)$

$$= \frac{1}{2} \sum_{n \in \mathbb{Z}} \sigma_{k-1}(|n|) \cdot q^{|n|}$$

Easy Check : if f is a modular form of wt k for Γ & g is wt l for Γ

then fg is wt $k+l$.

Fact $\bigoplus_{k \in \mathbb{Z}} M_k(\underbrace{SL_2(\mathbb{Z})}_{\text{level } 1}) = \mathbb{C}[E_4, E_6]$

q-expansion: $E_4 = 1 + 240 \cdot \sum_{n \geq 1} \sigma_3(n) \cdot q^n$

& $E_6 = 1 - 504 \cdot \sum_{n \geq 1} \sigma_5(n) \cdot q^n$. $\sigma_3(d)\,\sigma_3(n-d)$

Rmk:

$E_8 = 1 + 480 \sum_{n \geq 1} \sigma_7(n) q^n$

$\therefore E_8 = E_4^2 \Rightarrow$ funny relation between $\sigma_7(n)$ & $\sigma_3(d)$, $d < n$

For $SL_2(\mathbb{Z})$, only cusp is ∞. & we build our first cusp form Δ by $\Delta = \frac{E_4^3 - E_6^2}{1728} = q - 24q^2 + \cdots \in \mathbb{Z}[[q]]$

Remark: $\Gamma_1(N) \lhd \Gamma_0(N)$

$\Gamma_0(N) \longrightarrow (\mathbb{Z}/N\mathbb{Z})^\times$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto d$ is a gp homomorphism.

& hence $(\mathbb{Z}/N\mathbb{Z})^\times$ acts on $M_k(\Gamma_1(N))$ & $S_k(\Gamma_1(N))$

$d \rightsquigarrow$ lift to $\gamma_d \in \Gamma_0(N)$ $\quad f \longmapsto f|_k \gamma_d$

These are called the Diamond operators

Another easy remark : if $\Delta \subseteq \Gamma$ is finite index,

& $f \in M_k(\Gamma)$ then $f \in M_k(\Delta)$

OTOH, if $g \in M_k(\Delta)$ & we write $\Gamma = \coprod_{i=1}^{r} \Delta \gamma_i$

then $\sum_{i=1}^{r} g|\gamma_i \in M_k(\Gamma)$

Hecke operators are examples of such operator?