April 20. 2006. Thursday. Christophe Breuil

(2:30pm – 4:00pm)

(Lecture 4)

Aspects: Mod p

$GL_2(\mathbb{Q}_p)$ ⇝ modulo p local correspondence — weight part of Serre's Conjecture

Buzzard. Diamond. Jarvis. Gee.

weights

↓

mod p correspondence for $GL_2(\mathbb{Q}_{p^2})$?

Classification of smooth irred. admissible rep'n of $GL_n(\mathbb{Q}_{p^2}) / \overline{\mathbb{F}}_p$?

The $GL_2(\mathbb{Q}_p)$ - case

Classification: 1-dim'l rep'ns

principal series $\mathrm{Ind}_B^G \chi_1 \otimes \chi_2$, $\chi_1 \neq \chi_2$

the special representation: Steinberg $\otimes \chi$

Super singular representation

$$\frac{c\text{-}\mathrm{ind}_{KZ}^{G} \mathrm{Sym}^r \overline{\mathbb{F}}_p^2}{(T)} \otimes \chi \circ \det \qquad r \in \{0, \dots, p-1\}$$

$\dim_{\overline{\mathbb{F}}_p}(\mathrm{Sym}^r \overline{\mathbb{F}}_p^2) = r+1$

T: some Hecke operator

$$\left(\begin{array}{l} K = GL_2(\mathbb{Z}_p) \\ Z = \mathbb{Q}_p^\times \end{array} \right. \qquad \begin{array}{l} p \mapsto 1 \\ K \twoheadrightarrow GL_2(\mathbb{F}_p) \end{array}$$

$$\mathrm{End}_G\left(c\text{-}\mathrm{ind}_{KZ}^G \mathrm{Sym}^r \overline{\mathbb{F}}_p^2\right) = \overline{\mathbb{F}}_p[T]$$

ω = cycl. char. mod p.

$$\left(\mathrm{Ind}_B^G \chi_2 \otimes \chi_1 \omega^{-1}\right)^{ss} \oplus \left(\mathrm{Ind}_B^G \chi_1 \otimes \chi_2 \omega^{-1}\right)^{ss} \longleftrightarrow \bar\rho = \begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix}$$

$$\frac{c\text{-}\mathrm{ind}_{KZ}^G \mathrm{Sym}^r \overline{\mathbb{F}}_p^2}{(T)} \longleftrightarrow \bar\rho\Big|_{\text{Inertia}} \simeq \begin{pmatrix} \omega_2^{r+1} & 0 \\ 0 & \omega_2^{p(r+1)} \end{pmatrix}$$

$$\det(\bar\rho) \simeq \omega^{r+1}$$

weights

$$\left[\mathrm{Sym}^r, \mathrm{Sym}^{p-1-r} \otimes (\omega^r \circ \det)\right] \longleftrightarrow \bar\rho\Big|_I \simeq \begin{pmatrix} \omega_2^{r+1} & 0 \\ 0 & \omega_2^{p(r+1)} \end{pmatrix}$$

$$0 \le r \le p-1$$

$$\frac{c\text{-}\mathrm{ind}_{KZ}^G \mathrm{Sym}^r}{(T)} \simeq \frac{c\text{-}\mathrm{ind}_{KZ}^G \mathrm{Sym}^{p-1-r}}{(T)} \otimes \omega^r \circ \det$$

$GL_2(\mathbb{Q}_{p^2})$ $\quad \mathbb{Q}_{p^2}$ = quad. unram. ext'n of $\mathbb{Q}_p$ $\qquad \mathbb{F}_{p^2}^\times \hookrightarrow \overline{\mathbb{F}}_p^\times$

(s_0, s_1) stands for $\mathrm{Sym}^{s_0} \overline{\mathbb{F}}_p^2 \otimes \left(\mathrm{Sym}^{s_1} \overline{\mathbb{F}}_p^2\right)^{\mathrm{Frob}}$

$$\omega_2 : \mathrm{Gal}(\overline{\mathbb{Q}}_p / \mathbb{Q}_{p^2}) \to \overline{\mathbb{F}}_p^\times$$

$$\left[\begin{array}{ll} (r_0, r_1) & (r_0+1, p-2-r_1) \otimes \omega_2^{p(r_1+1)} \\ (p-1-r_0, p-3-r_1) \otimes \omega_2^{r_0+p(r_1+1)} & (p-2-r_0, r_1+1) \otimes \omega_2^{r_0+1+p} \end{array}\right] \longleftrightarrow \bar\rho\Big|_{\text{Iner.}} \simeq \begin{pmatrix} \omega_4^{r_0+1+p(r_1+1)} & 0 \\ 0 & \omega_4^{p^2(\quad)} \end{pmatrix}$$

$$\det \bar\rho \simeq \omega_2^{r_0+1+p(r_1+1)}$$

We would have hoped to have

FALSE

$$\left[\ \frac{c\text{-ind}(r_0, r_1)}{(T)} \simeq \frac{c\text{-ind}(r_0-1,\ p-2-r_1)}{(T)} \otimes (\omega_2^{p(r_1+1)} \circ \det) \simeq \dots\ \right]$$

<u>Thm</u> (Paškūnas) $\qquad G = GL_2(\mathbb{Q}_{p^2})$

(i) One has G-equiv. embeddings:

$$\begin{array}{c} c\text{-ind}(s_0-1,\ p-2-s_1)\otimes\omega_2^{p(s_1+1)} \\ \oplus \\ c\text{-ind}(p-2-s_0,\ s_1-1)\otimes\omega_2^{s_0+1} \end{array} \hookrightarrow \frac{c\text{-ind}(s_0, s_1)}{(T)} \quad \left[\begin{array}{l}(s_0, s_1) \neq (0,0) \\ \neq (p-1, p-1)\end{array}\right]$$

(ii) Denote by $\varphi(s_0, s_1)$ the cokernel,

then $\varphi(s_0, s_1) \simeq \varphi(p-1-s_0,\ p-1-s_1)\otimes\omega_2^{s_0+ps_1}$

<u>Thm</u> (Buzzard, Diamond, Emerton, Gee, ...)

The 4 above representations $\left(\frac{c\text{-ind}\,\sigma_i}{(T)},\ i\in\{1,\dots,4\}\right)$

admit at least one common irreducible adm. quotient.

Uses results of Gee + Schein + Paškūnas' <u>Thm</u>.

global modular $\bar\rho$ such that $\bar\rho|_{D_p} \simeq \begin{pmatrix}\omega_4^{r_0+1+p(r_1+1)} & 0 \\ 0 & \text{conj}\end{pmatrix}$

$$c\text{-ind}(r_0-1,\ p-1-r_1)\otimes\omega_2^{r_0+pr_1} \hookrightarrow \frac{c\text{-ind}(r_0, r_1)}{(T)} \longrightarrow \bar\rho \quad \text{part of cohomology}$$

$$\downarrow$$

$$\frac{c\text{-ind}(r_0-1,\ p-2-r_1)\otimes(\ )}{T} \qquad c\text{-ind}(p-1-r_0,\ p-1-r_1)\otimes(\ ) \quad \hookrightarrow \neq 0$$

$\longrightarrow$ this weight doesn't appear

A Local construction of common quotients

$\sigma_1 := (r_0, r_1)$

$\sigma_2 := (r_0-1,\ p-2-r_1)\otimes\omega_2^{p(r_1+1)}$

$\sigma_3 := (p-1-r_0,\ p-2-r_1)\otimes\omega_2^{r_0+p(r_1+1)} \qquad (p-2-r_0,\ r_1+1)\otimes\omega_2^{r_0+p(p\,\cdot)}$

σ = weight

$Inj(\sigma)$ = injective envelope of σ in the category of smooth rep'ns of K over $\overline{\mathbb{F}}_p$.

$GL_2(\mathbb{Z}_{p^2}/p^n)$: unique up to **non-unique** isomorphism essential extension of σ.

$$V := Inj\,\sigma_1 \oplus Inj\,\sigma_2 \oplus Inj\,\sigma_3 \oplus Inj\,\sigma_4 \circlearrowleft KZ.$$

I = Iwahori $\subseteq K$. $I = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K \;\middle|\; c \equiv 0\,(p) \right\}$

$I(1)$ = pro-p Iwahori $\subseteq I$. $I(1) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K \;\middle|\; \begin{matrix} a \equiv 1\,(p) \\ d \equiv 1\,(p) \end{matrix}\; c \equiv 0\,(p) \right\}$

χ_i = character by which I acts on $\sigma_i^{I(1)} = \overline{\mathbb{F}}_p \cdot v_i$

<u>Lemma 1</u> There is a (non-canonical) I-equiv. isom.

$$Inj(\sigma)\big|_I \simeq \bigoplus_{\substack{\chi : \text{char of } I \\ \text{s.t. } \sigma \text{ is a subquot} \\ \text{of } Ind_{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}}^{GL_2(\mathbb{F}_{p^2})} \chi}} Inj(\chi)$$

$Inj(\chi)$ → injective envelope of χ in the category of smooth rep'ns of $I / \overline{\mathbb{F}}_p$.

If χ = char of I

$$\underline{\chi}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \chi_1(a) \cdot \chi_2(d)$$

χ_{1r}

$$\underline{\chi}^s = \chi^s\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \chi_2(a) \cdot \chi_1(d)$$

$$\begin{aligned}
Inj\,\sigma_1\big|_I &\cong Inj\,\chi_1 \oplus Inj\,\chi_4^s \oplus Inj\,\chi_4 \oplus Inj\,\chi_1^s \\
Inj\,\sigma_2\big|_I &\cong Inj\,\chi_2 \oplus Inj\,\chi_1^s \oplus Inj\,\chi_1 \oplus Inj\,\chi_2^s \\
Inj\,\sigma_3\big|_I &\cong Inj\,\chi_3 \oplus Inj\,\chi_2^s \oplus Inj\,\chi_2 \oplus Inj\,\chi_3^s \\
Inj\,\sigma_4\big|_I &\cong Inj\,\chi_4 \oplus Inj\,\chi_3^s \oplus Inj\,\chi_3 \oplus Inj\,\chi_4^s
\end{aligned}$$

<u>Lemma 2</u> (Paškunas): Let N be the normalizer of I inside G

Let $\chi : I \longrightarrow \overline{\mathbb{F}}_p^\times$, $\chi \neq \chi^s$

Then one can extend the action of I on $\mathrm{Inj}\,\chi \oplus \mathrm{Inj}\,\chi^s$ to an action of N st $p \in \mathbb{Z}$ act trivially and $\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$ exchange $\mathrm{Inj}\,\chi$ and $\mathrm{Inj}\,\chi^s$.

Pf) Check that $(\mathrm{Inj}\,\chi)^{\pi} \rightsquigarrow g \in I$

$$N = \langle I\mathbb{Z},\ \pi = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} \rangle \quad \begin{matrix} \| \\ (\mathrm{Inj}\,\chi^s) \end{matrix} \qquad g * v := \pi g \pi^{-1} v$$

$$\chi^s = \chi^{\pi} \rightsquigarrow \mathrm{Ind}_{I\mathbb{Z}}^{N} (\mathrm{Inj}\,\chi)$$

$$W := \bigoplus_{\substack{8 \text{ pairs } (\chi_i, \chi_i^s) \\ \text{as above}}} (\mathrm{Inj}\,\chi_i \oplus \mathrm{Inj}\,\chi_i^s) \circlearrowleft N$$

Fix a $I\mathbb{Z}$-equivariant isomorphism

$$\phi : V \xrightarrow{\sim} W$$

non-canonical

Lemma 3 (Paskunas):

There is a unique smooth G-rep'n Ω such that

$\Omega|_{K\cdot\mathbb{Z}} \cong V$ and $\Omega|_N \cong W$

the diagram:

$$\begin{array}{ccc} \Omega & \xrightarrow{\sim} & V \\ {\scriptstyle id}\| & & \downarrow{\scriptstyle i}\ \phi \\ \Omega & \xrightarrow{\sim} & W \end{array}$$

commutes.

uniqueness is clear. $K\cdot\mathbb{Z}$ $\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$ generates G.

[existence is proved using coefficient system]

Lemma 3 $\Rightarrow$ Ω = G-rep'n

$$\mathrm{soc}_K(\Omega) = \sigma_1 \oplus \sigma_2 \oplus \sigma_3 \oplus \sigma_4 \qquad \underbrace{\left(\mathrm{soc}_K(\mathrm{Inj}\,\sigma)\right)}_{=\sigma}$$

$\mathrm{Soc}\,\pi$:= the subspace of π generated by all irred subrep'ns of π.

Let π be the G-subrep'n of Ω generated by $\mathrm{Soc}_K(\Omega)$.

Thm π is irreducible admissible supersingular and is a common quotient of c-ind σ_i ...

proof : • admissibility

$$\Pi^{I(1)} \subseteq \Omega^{I(1)} \simeq \underset{8\ \text{span}(\chi_i,\chi_i^s)}{\bigoplus} (\mathrm{Inj}\,\chi_i)^{I(1)} \oplus (\mathrm{Inj}\,\chi_i^s)^{I(1)}$$

$$\simeq \underset{8\ \text{parts}}{\bigoplus} \chi_i \oplus \chi_i^s \quad (16 \text{ characters over } \overline{\mathbb{F}}_p)$$

H = profinite group.

$H(1) \trianglelefteq H$ = normal pro-p subgp open. ρ = irred rep'n of $H/\overline{\mathbb{F}}_p$.

$(\mathrm{Inj}\,\rho)^{H(1)}$ = inj. envelope of ρ in the category of fin-dim $\overline{\mathbb{F}}_p$-rep'n of $H/H(1)$

$\circlearrowright$ $H/H(1)$ $\quad$ $H/H(1) \nearrow$

apply this to I, $I(1)$

• irreducibility

$\pi' \subseteq \pi$

$\quad\hookrightarrow$ non-zero G-subsp.

$0 \neq \mathrm{Soc}_K(\pi') \subseteq \mathrm{Soc}_K(\pi) = \sigma_1 \oplus \sigma_2 \oplus \sigma_3 \oplus \sigma_4$

maximal semi-simple submodule $\quad$ also distinct

Say $\sigma_3 \subseteq \mathrm{Soc}_K(\pi')$

$\Pi = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$. $\Pi \cdot v_3 \in \mathrm{Inj}\,\sigma_4$

$\langle K\ \Pi \cdot v_3 \rangle \subset \pi' \Rightarrow \sigma_4 \subset \pi'$ and irred.

• Supersingular : We know that $\Pi^{I(1)}$ is at least 8 dim'l

$(v_i, \Pi v_i)_{i=1,\dots,4} \in \pi$

as π is irred. adm. $\Rightarrow$ it is supersingular

because it can't be (principal series.
(special series