April 18. 2006. Tuesday Breuil C. (F.V)

(Lecture 3)

Aspect III : Drinfeld Spaces.

Steinberg (of $\mathcal{L}$-invariant)

$GL_2(\mathbb{Q}_p)$

$\boxed{\begin{array}{l}\text{locally analytic rep'ns} \\ \text{filtered module}\end{array}}$

V = abs. irred. rep'n of $Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ on 2-dim K-v. space. HT wts $(0, k-1)$

Banach spaces
Galois rep'ns) (φ, Γ)-module

$(\varphi, \Gamma) \rightsquigarrow B(V) \circlearrowleft B(\mathbb{Q}_p)$

$\downarrow$ extend to (conjecture)

$GL_2(\mathbb{Q}_p)$ Colmez is working on

V: De Rham

$\underset{\underset{\displaystyle Sym^{k-2}K^2}{\shortparallel}}{\rho(V)} \otimes_K \pi(V)$

The semi-stable (non-crystalline) case (Steinberg)

V $(0, k-1)$ abs. irred semi-stable non-crystalline

$\underset{\underset{\displaystyle D_{st}}{\shortparallel}}{D} = Ke_1 \oplus Ke_0$ $\quad \begin{cases} \varphi(e_1) = p^{-\frac{k-2}{2}} e_1 \\ \varphi(e_0) = p^{-\frac{k}{2}} e_0 \end{cases}$ $\quad \begin{cases} N(e_1) = e_0 \\ N(e_0) = 0 \end{cases}$

$$Fil^i D = \begin{cases} \text{all} & \text{if } i \le -(k-1) \\ K(e_1 + \mathcal{L} e_0) & \text{if } -(k-2) \le i \le 0 \\ 0 & \text{if } i \ge 1 \end{cases}$$ $\mathcal{L} \in K$ "$\mathcal{L}$-invariant"

$$\pi(V) = \text{Steinberg} \otimes |\det|^{\frac{k-2}{2}}$$

Σ_0 = p-adic "upper" half plane.

rigid analytic $\underset{\text{(quasi)}}{\text{Stein}}$ space $/\mathbb{Q}_p$

$\Sigma_0(\mathbb{C}_p) = \mathbb{C}_p \setminus \mathbb{Q}_p$

$$H^1_{dR}(\Sigma_0) = \frac{\Omega^1(\Sigma_0)}{d\mathcal{O}(\Sigma_0)} \simeq \check{\text{Sternberg}} \rightsquigarrow \text{algebraic dual}$$

$$\Omega^1(\Sigma_0) \xrightarrow{\sim} \{\text{Rigid analytic ftns on } \Sigma_0\}$$

$$f(z)\,dz \longmapsto f(z) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} f = \frac{ad-bc}{(bz+d)^2} \cdot f\left(\frac{az+c}{bz+d}\right)$$

wt 2 action

$$\mathcal{O}(k) = \left\{ \begin{array}{l}\text{rigid analytic ftns on } \Sigma_0 \\ \text{which are } \underbrace{K\text{-rational}}\end{array} \right\}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f := |ad-bc|^{-\frac{k-2}{2}} \cdot \frac{ad-bc}{(bz+d)^k} \cdot f\left(\frac{az+c}{bz+d}\right)$$

$$\underset{\text{affinoid}}{U} = \mathbb{P}^1(\mathbb{C}_p) \setminus \text{discs around } z_i$$

$$f\big|_U = \sum_{n=0}^{\infty} b_n z^n + \sum_i \sum_{n=1}^{\infty} \frac{b_{i,n}}{(z-z_i)^n}$$

$$\boxed{\underset{\text{Fréchet space}}{\mathcal{O}(k)} \simeq \varprojlim \underset{\text{Banach space}}{\mathcal{O}(k)_U}}$$

$\in K$

$$\log_{\mathcal{L}} : \mathbb{C}_p \setminus \{0\} \longrightarrow \mathbb{C}_p \quad = p\text{-adic logarithm}$$

$$\text{s.t. } \log_{\mathcal{L}}(p) := \mathcal{L}$$

Take $U =$ quasi-cpt affinoid as before.

$$\mathcal{O}(k,\mathcal{L})_U := \left\{ f : U \to \mathbb{C}_p,\ f = \sum_{n=0}^{\infty} b_n z^n + \sum_i \sum_{n=1}^{\infty} \frac{b_{i,n}}{(z-z_i)^n} + \sum_i \sum_{n=0}^{k-2} c_{i,n} z^n \cdot \log(z-z_i) \right\}$$

$$\mathcal{O}(k,\mathcal{L}) := \varprojlim_U \mathcal{O}(k,\mathcal{L})_U \qquad \text{Fréchet space}$$

$\circlearrowleft$ $GL_2(\mathbb{Q}_p)$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f = \frac{|ad-bc|^{\frac{k-2}{2}}}{(ad-bc)^{k-2}} (bz+d)^{k-2} \cdot f\left(\frac{az+c}{bz+d}\right)$$

$$\mathcal{O}(k,\mathcal{L}) \twoheadrightarrow \mathcal{O}(k) \qquad (GL_2(\mathbb{Q}_p)\text{-invariant})$$

$$f \longmapsto f^{(k-1)}$$

Thm : (i) $\mathcal{O}(k,\mathcal{L})^\vee$ (continuous dual) is locally analytic representation of $GL_2(\mathbb{Q}_p)$ (Schneider–Teitelbaum) that has 3 J-H factors which are :

$$\mathcal{O}(k)^\vee \begin{cases} \mathrm{Sym}^{k-2} K^2 \otimes \mathrm{Steinberg} \otimes |\det|^{\frac{k-2}{2}} & \rightsquigarrow \text{unique irred. sub-quotient inside } \mathcal{O}(k,\mathcal{L})^\vee \\ \left(\mathrm{Ind}_{B}^{GL_2} a^{k-1} \otimes d^{-1}\right)^{an} \otimes |\det|^{\frac{k-2}{2}} & \end{cases}$$

$B = \begin{pmatrix} a & * \\ 0 & d \end{pmatrix}$

$\uparrow$ Morita, Schneider, Teitelbaum

$$\mathrm{Sym}^{k-2} K^2 \otimes |\det|^{\frac{k-2}{2}} \rightsquigarrow \text{unique irred quotient}$$

(ii) The universal unitary completion of $\dfrac{\mathcal{O}(k)^\vee}{\mathrm{Sym}^{k-2} \otimes \mathrm{St} \otimes |\cdot|^{\frac{k-2}{2}}}$

is isomorphic to the Banach space of functions $f: \mathbb{Q}_p \to K$ which are $C^{\frac{k-2}{2}}$ in restriction to $\mathbb{Z}_p$

$$\left(f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n}, \quad n^{\frac{k-2}{2}} |a_n| \to 0 \text{ as } n \to \infty \right)$$

$+$ $x^{k-2} f\left(\frac{1}{x}\right)$ extends to $\mathbb{Z}_p$ as a f'tn of class $C^{\frac{k-2}{2}}$

modulo polynomials of degree $\le k-2$.

$\boxed{k>2}$ (iii) The univ. unitary completion of $\dfrac{\left(\mathrm{Ind}_B^G \cdots\right)^{an} \otimes |\cdot|^{\frac{k-2}{2}}}{\mathrm{Sym}^{k-2} K^2 \otimes |\cdot|^{\frac{k-2}{2}}} = 0$

(iv) The univ. unitary completion of $\mathcal{O}(k,\mathcal{L})^\vee$ is isomorphic to the quotient :

φ-Γ module, log-differential

$$B(V) := \frac{\text{univ. unitary completion of } \left(\mathrm{Sym}^{k-2} \otimes \mathrm{Stein}\right) \otimes |\cdot|^{\frac{k-2}{2}}}{\left\{ \begin{array}{l} \text{closure of the subsp. of } f \text{ of the form:} \\ f(x) = \sum_{i \in I} \lambda_i (x-a_i)^{n_i} \log_{\mathcal{L}}(x-a_i) \text{ where } \begin{cases} \frac{k-2}{2} < n_i \le k-2 \\ \deg\left(\sum_{i\in I} \lambda_i (x-a_i)^{n_i}\right) < \frac{k-2}{2} \end{cases} \end{array} \right\}}$$

$\nearrow P(V) \otimes \Pi(V)$

$\circlearrowleft GL_2(\mathbb{Q}_p)$

$$\mathcal{O}(k)^\vee \hookrightarrow \mathcal{O}(k\ I)^\vee \longrightarrow \mathrm{Sym}^{k-2} \otimes 1\cdot 1$$

$$\widehat{\mathcal{O}(k)}^\vee \twoheadrightarrow \widehat{\mathcal{O}(k\ I)}^\vee$$

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universal unitary completion (Emerton)

Let W = locally convex topological K-vector space.

+ conti. $GL_2(\mathbb{Q}_p)$-action.

<u>Functor</u>. $GL_2(\mathbb{Q}_p)$ - unitary Banach spaces $\longrightarrow$ **Sets**.

$$B \longmapsto \mathrm{Hom}_{GL_2(\mathbb{Q}_p)}(W, B)$$

If representable, then the corresponding representing object.

= universal unitary completion of $W = \widehat{W}$.

$W \longrightarrow \widehat{W}$ image is always dense

but $\widehat{W}$ can be 0.

There is a sufficient condition so that $\widehat{W}$ exists at least

if W is a locally analytic rep'n of $GL_2(\mathbb{Q}_p)$.

$W^\vee$ admits a continuous semi-norm $q : W^\vee \to \mathbb{R}$

$\|$ such that the collection $(gq,\ g \in GL_2(\mathbb{Q}_p))$

Fréchet space gives back the Fréchet topology on $W^\vee$.

In that case:

$$\widehat{W} = \text{dual of } \underbrace{\{ v \in W^\vee \mid q(gv) \leq 1,\ \forall g \}}_{\text{cpct module}} \otimes K$$

$\uparrow$

Banach space

Ex.

$$W^\vee = \left| \begin{array}{l} \mathcal{O}(\mathfrak{k}) \\ \mathcal{O}(\mathfrak{k}, \mathcal{L}) \end{array} \right.$$

, choose U affinoid such that $(g \cdot U)_g$ covers Σ_U

$$\gamma : \mathcal{O}(\mathfrak{k}, \mathcal{L}) \to \mathcal{O}(\mathfrak{k}, \mathcal{L})_U \xrightarrow[\text{norm on } \mathcal{O}(\mathfrak{k},\mathcal{L})_U]{} \mathbb{R}$$

A supercuspidal case : (Ongoing Project with **M. Strauch**)

V abs irred pot-crystalline 2-dim/K repl'n of $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$

$HT = (0, 1)$

V becomes crystalline over $Gal(\overline{\mathbb{Q}}_p/F)$

For simplicity $\boxed{K = \mathbb{Q}_p}$ $\qquad F (= L') = \mathbb{Q}_{p^2}(\sqrt[p^2-1]{-p})$

$$D = \overbrace{\mathbb{Q}_{p^2}}^{= F_0} e_\chi \oplus \mathbb{Q}_{p^2} e_{\chi^p}$$

$\chi : Gal(F/\mathbb{Q}_{p^2}) \to \mathbb{Q}_{p^2}^\times$, $\chi \neq \chi^p$

$$\begin{cases} \varphi(e_\chi) = e_{\chi^p} \\ \varphi(e_{\chi^p}) = -\frac{1}{p} e_\chi \end{cases} \qquad \begin{cases} g(e_\chi) = \chi(g) \cdot e_\chi \\ g(e_{\chi^p}) = \chi^p(g)\, e_{\chi^p} \end{cases} \qquad g \in Gal(F/\mathbb{Q}_{p^2})$$

$$\begin{cases} g_\varphi(e_\chi) = e_\chi \\ g_\varphi(e_{\chi^p}) = e_{\chi^p} \end{cases} \qquad g_\varphi \in Gal(F/\mathbb{Q}_p)$$

$$Fil^i(D \otimes_{\mathbb{Q}_{p^2}} F) = \begin{cases} \text{all} & \text{if } i \le -1 \\ F(\varpi_\chi a e_\chi + b e_{\chi^p}) & \text{if } i = 0 \qquad (a,b) \in \mathbb{P}^1(\mathbb{Q}_p) \\ 0 & \text{if } i \ge 1 \end{cases}$$

$\varpi_\chi \in F$ = smallest power of $\sqrt[p^2-1]{p}$

so that $g(\varpi_\chi)$

$= \chi(g)^{p-1} \varpi_\chi$, $g \in Gal(F/\mathbb{Q}_{p^2})$

$\rho(V)$ = trivial.

$$\pi(V) = \text{c-ind}_{GL_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{GL_2(\mathbb{Q}_p)} DL(\chi) \qquad DL(\chi) = \text{supercuspidal } \mathbb{Q}_p\text{-repn of } GL_n(\mathbb{Z}_p) \text{ associated to } \chi$$

First covering Σ_1 of Σ_0

$$\underset{\text{quaternion alg}}{D^\times} \curvearrowright \Sigma_1^{(*)} / \hat{\mathbb{Q}}_p^{unr}$$

$$\Sigma_1^{(*)} / \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}^{\mathbb{Z}} \cong \Sigma_1^{(0)}/_{\mathbb{Q}_{p^2}} \amalg \Sigma_1^{(1)}/_{\mathbb{Q}_{p^2}}$$

$\Sigma_1 \circlearrowleft GL_2(\mathbb{Q}_p)$

$\circlearrowleft$

$\mathcal{O}_D^\times$

$g \circ d = d \circ g$ if $val(\det(g))$ is even

otherwise $g \circ d = \bar{d} \circ g$

$F_F := \varpi_D \circ \alpha$

use it as a descent data to define $\Sigma_1^{(0)}/_{\mathbb{Q}_p} =: \Sigma_1$

$$\left.\begin{matrix}\Sigma_1 \\ \downarrow \\ \Sigma_0\end{matrix}\right) \text{ Galois covering} \qquad \text{Galois gp} \cong \mathbb{F}_{p^2}^\times \cong \left(\mathcal{O}_D/m_D\right)^\times$$

Teitelbaum = a semi-stable formal model of Σ_1 over $\mathcal{O}_F$

(S Venn)

Thm. $$\underbrace{H^1_{(HK)}}_{\text{Hyodo-Kato}}\left(\Sigma_1 \times_{\mathbb{Q}_p} F\right)^{\text{action of } \mathbb{F}_{p^2}^\times \leftarrow (\chi^{-1}+\chi^{-p})} \xrightarrow{\sim} \left(\mathrm{Ind}_{GL_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{GL_2(\mathbb{Q}_p)} DL(\chi)^\vee\right) \otimes_{\mathbb{Q}_p} D^\vee \qquad \boxed{\chi \neq \chi^p}$$

Compatible with $GL_2(\mathbb{Q}_p)$, φ, $Gal(F/\mathbb{Q}_p)$

$\hookrightarrow$ Grosse-Klönne

$$H^1_{\log\text{-}cris}\left(\begin{matrix}\text{special} \\ \text{fiber}\end{matrix} / W(\mathbb{F}_{p^2}) = \mathbb{Z}_{p^2}\right) \otimes \mathbb{Q}_{p^2}$$

$F_0 \cong$, $\circlearrowleft \varphi$

$$\underbrace{\Omega^1(\Sigma_1)^{\chi^{-1}+\chi^{p}}}_{\substack{\parallel \\ \Omega^1(\Sigma_1) \cap \Omega^1(\Sigma_1 \otimes_{\mathbb{Q}_p} F)^{(\chi^{-1}+\chi^{-p})}}} \hookrightarrow \Omega(\Sigma_1 \times F) \twoheadrightarrow H^1_{dR}(\Sigma_1 \times F)^{\chi^{-1}+\chi^{-p}} \xrightarrow{\sim} \mathrm{Ind}\, DL(\chi)^\vee \otimes_{\mathbb{Q}_p} D_F^\vee$$

R

$(a,b) \in \mathbb{P}^1(\mathbb{Q}_p)$ $\quad \mathrm{Fil}^1 D_F^\vee = F(\varpi_\chi a e_{\chi_1} + b e_{\chi^p})$

$$\Pi_{\chi,(a,b)} := \rho^{-1}\left(\mathrm{Ind}\, DL(\chi)^\vee \otimes \mathrm{Fil}^1 D_F^\vee\right)$$

$B(V)$ = universal unitary completion of $\Pi^\vee_{\chi,(a,b)}$

$\boxed{\text{Prop}}$. (i) $\Pi^\vee_{\chi,(a,b)}$ is a loc. analy rep'n of $GL_2(\mathbb{Q}_p)/\mathbb{Q}_p$ that is an extension:

$$0 \to c\text{-}\mathrm{Ind}\, DL(\chi) \to \Pi^\vee_{\chi,(a,b)} \to \left(\mathcal{O}(\Sigma_1)^\vee\right)^{\chi+\chi^p} \to 0$$

isotypic part

Deligne-Lusztig curve

(ii) In $\mathrm{Ext}^1_{an}\left(\mathcal{O}(\Sigma_1)^{\vee\,\chi+\chi^p}, c\text{-}\mathrm{Ind}\, DL(\chi)\right)$

$$[\Pi^\vee_{\chi,(a,b)}] = a[\Pi^\vee_{\chi,(1,0)}] + b[\Pi^\vee_{\chi,(0,1)}]$$

(iii) $\Pi^\vee_{\chi,(a,b)}$ admits a universal unitary completion which is also a completion of $c\text{-}\mathrm{Ind}\, DL(\chi)$.

Universal unitary completion of $c\text{-}\mathrm{Ind}\, DL(\chi)$

$=$ functions $f : GL_2(\mathbb{Q}_p) \to DL(\chi)$
s.t. $f(kg) = k \cdot f(g)$ $\quad (k \in K\mathbb{Z})$
$+$ $f(g)$ tends to p-adically 0 when $g \to \infty$

Do we have something like this?

$$\widehat{\Pi^\vee_{\chi,(a,b)}} = \frac{(c\text{-}\mathrm{Ind}\, DL(\chi))^\wedge}{\left(\text{closure of some explicit subspace depending on } (a,b)\right)} \;? = B(V)$$