Jeudi 13/04/06

Aspects of the p-adic local Langlands programme

Aspect II : (φ,Γ) - modules

A very impatant aspects of p-adic Langlands is the link with (φ,Γ)-modules, that has first been found by Colmez. I will explain in this talk how it can be used to prove cases of the previous con--jecture (see aspect I) for $GL_2(\mathbb{Q}_p)$. This aspect is in full development.

Quick reminder on (φ,Γ)-modules.

L. Berger has already given a talk on (φ,Γ)-modules here, so I will only recall some basic facts. I fix a compatible system of primitive p^n roots of 1, $(\zeta_{p^n})_{n\geq 0}$.

Let V be a continuous represent: of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ on a finite dimensional K-vector space. Then one can associate to V in a functorial way a free $\left(K\otimes_{\mathcal{O}_K}\mathcal{O}_K[[X]][\frac{1}{X}]^{\wedge}\right)$-module of $\mathrm{rk} = \dim_K V$, usually denoted $D(V)$, together with $\varphi: D(V)\longrightarrow D(V)$

$\gamma: D(V)\longrightarrow D(V)$, $\gamma\in\Gamma = \mathrm{Gal}\left(\mathbb{Q}_p(\zeta_{p^n})_n/\mathbb{Q}_p\right)$

φ is injective and semi-linear $\quad\left(\varphi(Xv) = ((1+X)^p-1)\varphi(v)\right)$

γ is bijective and semi-linear $\quad\left(\gamma(Xv) = ((1+X)^{\chi(\gamma)}-1)\gamma(v)\right)$.

Any $v\in D(V)$ can be written uniquely $\quad v = \sum_{i=0}^{p-1}(1+X)^i\varphi(v_i)\quad$ for some $v_i\in D(V)$ and one can define $\psi: D(V)\twoheadrightarrow D(V)$, $\psi(v) := v_0$.

Then $\psi\varphi = \mathrm{Id}$. There is a natural "weak" topology on $D(V)$ defined as follows. As $D(V)\simeq\left(K\otimes_{\mathcal{O}_K}\mathcal{O}_K[[X]][\frac{1}{X}]^{\wedge}\right)^n$, it is enough to define this topology on $K \otimes_{\mathcal{O}_K} \mathcal{O}_K[[X]][\frac{1}{X}]^\wedge$ (this will be independent on the choice of an isomorphism). Writing $K \otimes_{\mathcal{O}_K} \mathcal{O}_K[[X]][\frac{1}{X}]^\wedge = \bigcup_{n \geq 0} \frac{1}{\pi_K^n} \mathcal{O}_K[[X]][\frac{1}{X}]^\wedge$ and taking the inductive limit topology, it is enough to define it on $\mathcal{O}_K[[X]][\frac{1}{X}]^\wedge$. We define it by declaring that $\left(\pi_K^n \cdot \mathcal{O}_K[[X]][\frac{1}{X}]^\wedge + X^m \mathcal{O}_K[[X]]\right)_{n,m \geq 0}$ is a basis of neighbourhoods of 0.

In particular, a sequence $(v_i)_i$ of elements of $K \otimes_{\mathcal{O}_K} \mathcal{O}_K[[X]][\frac{1}{X}]^\wedge$ is bounded if there is n_0 s.t. $(\pi_K^{n_0} v_i)_i \in \mathcal{O}_K[[X]][\frac{1}{X}]^\wedge$ and if $(\pi_K^{n_0} v_i)_i$ is such that, $\forall n \geq 0, \exists m \in \mathbb{Z} \mid \pi_K^{n_0} v_i \in \pi_K^n \mathcal{O}_K[[X]][\frac{1}{X}]^\wedge + \frac{1}{X^m} \mathcal{O}_K[[X]]$ $(\forall i)$.

From this, we easily deduce what it means for a sequence $(v_i)_i$ of elements of $D(V)$ to be bounded.

<u>The basic recipee.</u>

I want to explain here the definition of a functor due to Colmez:

$$\left\{ (\varphi,\Gamma)\text{-modules} \right\} \longrightarrow \left\{ \begin{matrix} \text{continuous unitary representations of} \\ \begin{pmatrix} 1 & \mathbb{Q}_p \\ 0 & \mathbb{Q}_p^\times \end{pmatrix} \text{ on } K\text{-Banach spaces} \end{matrix} \right\}$$

$$D \longmapsto B(D)$$

recall that an object of the R.H.S. is a K-Banach space $B(D)$ together with a continuous map $\begin{pmatrix} 1 & \mathbb{Q}_p \\ 0 & \mathbb{Q}_p^\times \end{pmatrix} \times B(D) \longrightarrow B(D)$ such that there exists a norm $\| \ \|$ on $B(D)$ (giving the Banach topology) satisfying $\|g \cdot v\| = \|v\|$ $\forall g \in \begin{pmatrix} 1 & \mathbb{Q}_p \\ 0 & \mathbb{Q}_p^\times \end{pmatrix}$, $\forall v \in B(D)$.

Start from a (φ,Γ)-module D and define:

$$B^\vee(D) := \left\{ \text{bounded sequences in } \varprojlim_{\psi} D \right\}$$

by which I mean sequences $(v_i)_{i\in\mathbb{Z}_{\geq 0}}$ such that $\begin{cases} v_i \in D \\ \psi(v_i) = v_{i-1} \\ (v_i) \text{ is bounded} \end{cases}$.

We equip $B^\vee(D)$ with the projective limit topology, the topology on D being the above weak topology.

Then define an action of $\begin{pmatrix} 1 & \mathbb{Q}_p \\ 0 & \mathbb{Q}_p^\times \end{pmatrix}$ on $B^\vee(D)$ as follows:

- $\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ acts as $(v_i)_i \longmapsto (\psi(v_i))_i$
- $\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$ acts as $(v_i)_i \longmapsto (\gamma_a v_i)_i$

  $a \in \mathbb{Z}_p^\times$

  where $\Gamma \xrightarrow[\varepsilon^{-1}]{\sim} \mathbb{Z}_p^\times$, ε = p-adic cyclo

  $\gamma_a \longmapsto a$
- $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$ acts as $(v_i)_i \longmapsto \left(\varphi^i\left((1+X)^z\right) v_i \right)_i$

  $z \in \mathbb{Z}_p$

(this extends uniquely in an action of $\begin{pmatrix} 1 & \mathbb{Q}_p \\ 0 & \mathbb{Q}_p^\times \end{pmatrix}$).

<u>Define:</u> $B(D) = \left(B^\vee(D)\right)^\vee$ (continuous functions $B^\vee(D) \to K$)

$$= \mathrm{Hom}^c_{\mathcal{O}_K}\left(B^\vee(D^\circ), K \right)$$

endowed with the topology $\|f\| := \sup_{v^\vee \in B^\vee(D^\circ)} |f(v^\vee)|$

where D° is an $\mathcal{O}_K[[X]][\frac{1}{X}]^\wedge$-lattice inside D preserved by φ and Γ and where $B^\vee(D^\circ)$ is defined with D° exactly as $B^\vee(D)$ with D.

As $B^\vee(D^\circ)$ (with the weak topology) is <u>compact</u>, the above $\|f\|$ is well defined. The action of $\begin{pmatrix} 1 & \mathbb{Q}_p \\ 0 & \mathbb{Q}_p^\times \end{pmatrix}$ on $B^\vee(D)$ naturally

(because one can use $D^\sharp(T)$, so it is not a trivial fact) defines an action on $\mathrm{Hom}_K^{c}(B^\vee(D), K)$ by $g\cdot f := f(g^{-1}\cdot)$ and one can prove (I am not going to do this here!) that:

$$\begin{pmatrix}1 & \mathbb{Q}_p\\ 0 & \mathbb{Q}_p^\times\end{pmatrix} \times B(D) \longrightarrow B(D) \quad \text{is continuous.}$$

Note that $B(D)$ is obviously a unitary Banach space for this action, as $B^\vee(D^0)$ is preserved by $\begin{pmatrix}1 & \mathbb{Q}_p\\ 0 & \mathbb{Q}_p^\times\end{pmatrix}$ (because it is stable by φ, Γ).

<u>p-adic Langlands</u>

Composing with the functor

$$\left\{\begin{matrix}\subseteq \text{ repres. of}\\ \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)\\ \text{on } K\text{-v.s.}\end{matrix}\right\} \longrightarrow \{(\varphi,\Gamma)\text{-modules}\}, \qquad V \longmapsto D(V)$$

one gets like this a functor:

$$\left\{\begin{matrix}\subseteq \text{ repres. of}\\ \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)\\ \text{on } K\text{-v.s.}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{continuous unitary repr.}\\ \text{of } \begin{pmatrix}1 & \mathbb{Q}_p\\ 0 & \mathbb{Q}_p^\times\end{pmatrix} \text{ on } K\text{-Banach}\\ \text{spaces}\end{matrix}\right\}$$

Note that there is no restriction on the dimension of the represent: of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ here (apart from being finite).

We denote this functor $V \longmapsto B(V)$.

<u>Lemma:</u> If V is abs. irreducible of dim >1, then $B(V)$ is topologically irreducible as a represent° of $\begin{pmatrix}1 & \mathbb{Q}_p\\ 0 & \mathbb{Q}_p^\times\end{pmatrix}$.

<u>Sketch of proof:</u> This proof is due to Colmez. We prove that $B(V)^\vee$ is top. irreducible. The result from (φ,Γ)-module we use is the following:

> Assume V is abs. irreducible and non-abelian ($\Rightarrow$ dim >1), then there is a unique non-zero K-vector subspace $D^\#(V) \subseteq D(V)$ containing an $\mathcal{O}_K[[X]]$-submodule $D^{\#,0}$ that is bounded, preserved by ψ and Γ with ψ surjective, and generating (over $\mathcal{O}_K[[X]][\frac{1}{X}]^\wedge$).

[recall bounded means $D^{\#,0}/p^n$ is of f.t. over $\mathcal{O}_K/p^n[[X]]$ for all n]

Let T be a Galois lattice and $0 \neq B \subseteq \left(\varprojlim_\psi D(V)\right)^b = \left(\varprojlim_\psi D(T)\right)^b \otimes_{\mathcal{O}_K} K$ closed and stable under $\begin{pmatrix}1 & \mathbb{Q}_p\\ 0 & \mathbb{Q}_p^\times\end{pmatrix}$. Let:

$$M^\circ := \left\{ v \in D(T) \;\middle|\; \exists\, (v_i)_{i\in\mathbb{Z}_{\geq 0}} \in B \cap \left(\varprojlim_{\psi} D(T)\right)^b \text{ s.t. } v_0 = v \right\} \neq 0$$

Then it is easy to check that $\psi : M^\circ \to M^\circ$ is defined and surjective, and that $B \cap \left(\varprojlim_\psi D(T)\right)^b = \left(\varprojlim_\psi M^\circ\right)^b \Rightarrow B = \left(\varprojlim_\psi M^\circ\right)^b \otimes K$. One also proves that M° is a bounded $\mathcal{O}_K[[X]]$-module (as $M^\circ \subset D^\#(V)$) $\Rightarrow M^\circ \otimes K = D^\#(V)$. Thus $B = \left(\varprojlim_\psi M^\circ\right)^b \otimes K = \left(\varprojlim_\psi M^\circ \otimes K\right)^b$

$$= \left(\varprojlim_\psi D^\#(V)\right)^b \overset{(M^\circ\subset D^\#(T)) \text{ actually}}{=} \left(\varprojlim_\psi D(V)\right)^b. \quad \square$$

In general, one cannot extend the action of $\begin{pmatrix} 1 & \mathbb{Q}_p \\ 0 & \mathbb{Q}_p^\times \end{pmatrix}$ on $B(V)$ to an action of $GL_2(\mathbb{Q}_p)$. However:

<u>Hope 1:</u> | If V is abs. irreducible of dim 2, then the action extends uniquely to $GL_2(\mathbb{Q}_p)$.

<u>Hope 2:</u> | If V moreover is de Rham $\underbrace{}_{\text{with distinct HT weights}}$, then $B(V)$ with the action extended is a completion of $\rho(V) \otimes_K \pi(V)$, where $\rho(V) = \mathrm{Sym}^{i_2 - i_1 - 1} K^2 \otimes \det^{i_1}$ ($\text{HT weights}(V) = i_1 < i_2$) and where $\pi(V)$ is defined as follows:

$$\left(\varphi, N, \mathrm{Gal}(L/\mathbb{Q}_p), D\right) \xmapsto{\text{WD + F-ss}} (r, N, \underbrace{V}_{\text{not same } V!}) \longleftrightarrow \pi(V) \quad \left(\begin{matrix}\text{previous}\\ \text{talk}\end{matrix}\right).$$

(here, L is a Galois extension of $\mathbb{Q}_p$ over which V becomes semi-stable).

It seems that Colmez, using ideas of Kisin, is close to proving Hope 1.

<u>The case $\pi(V)$ = principal serie (and V abs. irred. of dim. 2).</u>

This case is the most simple case, as $(r, \underset{0}{\underset{\shortparallel}{N}})$ completely determines the irreducible Galois representation V. Note that $N = 0$ and L is abelian tot. ramified. Moreover, we will exclude the case where φ is not semi-simple on D (the method below doesn't work here).

<u>Thm</u> (Berger - B.) | Under the above assumptions ($\pi(V)$ = PS and φ semi-simple), one can extend the action of $\begin{pmatrix} 1 & \mathbb{Q}_p \\ 0 & \mathbb{Q}_p^\times \end{pmatrix}$ on $B(V)$ to an action of $GL_2(\mathbb{Q}_p)$

such that $B(V)$ is a unitary completion of $\rho(V)\otimes_K \Pi(V)$. Moreover, there is no other non-zero unitary completion of $\rho(V)\otimes_K \Pi(V)$. ⑥

rk: Note that this proves (almost) the conj. of Aspect I in the case $GL_2(\mathbb{Q}_p)$ and Π = P.S.

I give now the main steps of the proof.

Twisting and enlarging K if necessary, one can assume:

$k\geq 2$, $\alpha,\beta:\mathbb{Q}_p^\times \to K^\times$ smooth characters, distinct, trivial on $1+p^n\mathbb{Z}_p$ for $n\geq 1$

let $\alpha_p := \alpha(p)^{-1}$, $\beta_p := \beta(p)^{-1}$

- $D = Ke_\alpha \oplus Ke_\beta$ $\quad \begin{cases}\varphi(e_\alpha)=\alpha_p^{-1}e_\alpha\\ \varphi(e_\beta)=\beta_p^{-1}e_\beta\end{cases}$ $\quad \begin{cases}\gamma(e_\alpha)=\alpha(\varepsilon(\gamma))\,e_\alpha\\ \gamma(e_\beta)=\beta(\varepsilon(\gamma))\,e_\beta\end{cases}$ $\quad \gamma\in\Gamma := Gal(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p) \overset{\wr}{\downarrow}\ \varepsilon = \text{cycl. char.}\ \ \mathbb{Z}_p^\times$

$0\leq val(\alpha_p) < k-1$
$0\leq val(\beta_p) < k-1$
$val(\alpha_p)+val(\beta_p) = k-1$

$$Fil^i\left(\mathbb{Q}_p(\zeta_{p^n})\otimes_{\mathbb{Q}_p} D\right) = \begin{cases}\text{all} & \text{if } i\leq -(k-1)\\ \mathbb{Q}_p(\zeta_{p^n})\otimes_{\mathbb{Q}_p} K\cdot\left(e_\alpha + G(\beta\alpha^{-1})e_\beta\right) & \text{if } -(k-2)\leq i\leq 0\\ 0 & \text{if } i\geq 1.\end{cases}$$

Here $G(\beta\alpha^{-1})$ is the Gauss sum associated to $\beta\alpha^{-1}$, $G(\beta\alpha^{-1})\in\left(\mathbb{Q}_p(\zeta_{p^n})\otimes_{\mathbb{Q}_p} K\right)^\times$

$\left[\ G(\eta) = \sum_{\gamma\in\Gamma/\Gamma_n} \gamma(\zeta_{p^n})\otimes\eta^{-1}(\gamma)\right.$ if η is ramified, $\left. G(\eta)=1 \text{ if } \eta \text{ is unramified}\ \right]$

$\Gamma/\Gamma_n \simeq \mathbb{Z}_p^\times/1+p^n\mathbb{Z}_p$

$\lfloor\!\!\to$ [since one can rescale e_β in Ke_β, one can fix a compatible system of primitive (ζ_{p^n}) as in beginning]

- $$\rho\otimes_K\Pi = Sym^{k-2}K^2\otimes_K\left(ind_{\begin{pmatrix}* & *\\ 0 & *\end{pmatrix}}^{GL_2(\mathbb{Q}_p)}\alpha\otimes\beta|\,|^{-1}\right)$$

We assume Π generic in the sequel for simplicity.

To define an action of $GL_2(\mathbb{Q}_p)$ on $B(V)$, one has to proceed in 2 steps.

Step 1: describe the completion of $\rho\otimes_K\Pi$ with respect to any generating $\mathcal{O}_K[GL_2(\mathbb{Q}_p)]$-submodule of finite type.

Step 2: Show that $B(V)$ is actually isomorphic to this completion.

In step 1, one has an action of $GL_2(\mathbb{Q}_p)$, but one doesn't know if the completion is zero or not. In step 2, one has a non-zero space, but only with an action of $\begin{pmatrix} 1 & \mathbb{Q}_p \\ 0 & \mathbb{Q}_p^\times \end{pmatrix}$. So step 1 + 2 give the result, from which it is easy to deduce that $\rho \otimes \pi$ doesn't have any other non-zero unitary completion (using that $B(V)$ is irreducible and also admissible).

Step 1: We compute the dual of this completion. For this, the method is to fix any $\mathcal{O}_K[GL_2(\mathbb{Q}_p)]$-submodule of finite type and generating, call it M, and to compute which linear forms $\mu : \rho \otimes_K \pi \longrightarrow K$ satisfie $\|\mu(m)\| \leq 1$, $\forall m \in M$. Writing $\mathrm{Sym}^{k-2} K^2 \simeq \bigoplus_{j=0}^{k-2} K z^j$, one gets the following: $\forall a \in \mathbb{Q}_p$, $\forall j \in \{0, \dots, k-2\}$, $\forall n \in \mathbb{Z}$:

(i) $$\int_{a+p^n\mathbb{Z}_p} (z-a)^j \, d\mu(z) \in C_M \, p^{n(j - \mathrm{val}(\alpha_p))} \mathcal{O}_K$$ (C_M is some constant that we don't care)

(ii) $$\int_{\mathbb{Q}_p - (a+p^n\mathbb{Z}_p)} \beta\alpha^{-1}(z-a)\, |z-a|^{-1} (z-a)^{k-2-j} \, d\mu(z) \in C_M \, p^{n(\mathrm{val}(\alpha_p) - j)} \mathcal{O}_K$$

(i) means that μ is tempered of order $\leq \mathrm{val}(\alpha_p)$

(ii) means that μ is "tempered at ∞" of order $\leq \mathrm{val}(\alpha_p)$.

Working out more closely (i) and (ii), and using Amice-Vélu-Vishik, one finds that (i) + (ii) $\Longleftrightarrow$ μ extends to a continuous linear form on $B(\alpha)/L(\alpha)$ where: transparency for $B(\alpha)/L(\alpha)$.

Hence the completion of $\rho \otimes_K \pi \simeq B(\alpha)/L(\alpha) \simeq B(\beta)/L(\beta)$

this isomorphism extends the usual intertwining:

$\hookrightarrow$ same proof, but choose β instead of α

$$\rho \otimes \mathrm{ind}\, \alpha \otimes \beta |\,|^{-1} \xrightarrow{\sim} \rho \otimes \mathrm{ind}\, \beta \otimes \alpha |\,|^{-1}$$

- For $r \in \mathbb{R}_{\geq 0}$, a function $f: \mathbb{Z}_p \to K$ is C^r if the Mahler expansion $f(z) = \sum_{n=0}^{\infty} a_n \binom{z}{n}$ satisfies

$$n^r |a_n| \xrightarrow[n \to \infty]{} 0$$

$\rightsquigarrow C^r(\mathbb{Z}_p, K)$: Banach space [illegible]

- $B(\alpha) :=$ Banach space of functions $f: \mathbb{Q}_p \to K$ s.t. $f|_{\mathbb{Z}_p}$ is $C^{val(\alpha_p)}$ and $\beta\alpha^{-1}(z)|z|^{-1} z^{k-2} f(\frac{1}{z})|_{\mathbb{Z}_p - \{0\}}$ extends as a function $C^{val(\alpha_p)}$

- $\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f\right)(z) = \alpha(ad-bc)\beta\alpha^{-1}(-cz+a)|-cz+a|^{-1}(-cz+a)^{k-2} \cdot f\left(\frac{dz-b}{-cz+a}\right)$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Q}_p)$

- $L(\alpha) =$ closure of subspace generated by

$z \longmapsto z^j$

$z \longmapsto \beta\alpha^{-1}(z-a)|z-a|^{-1}(z-a)^{k-2-j}$

for $a \in \mathbb{Q}_p$ and $0 \leq j < val(\alpha_p)$

Then $B(\alpha)/L(\alpha) =$ unitary $GL_2(\mathbb{Q}_p)$-Banach space

Step 2: One proves:

$$\left(B(\alpha)/L(\alpha)\right)^\vee \xrightarrow{\sim} \left(\varprojlim_{\psi} D(V)\right)^b$$

$$\{\mu \in B(\alpha)^\vee \mid \langle \mu, L(\alpha)\rangle = 0\}$$

This map is essentially | p-adic Fourier transform | Amice transform.

Let $\mathcal{R}_K^+ = \left\{\sum_{n\geq 0} a_n X^n \in K[[X]] \text{ converging on the open unit disc}\right\}$, the map $\mu \longmapsto \sum_{n=0}^{+\infty} \langle \mu, \binom{z}{n}\rangle X^n$ induces an isomorphism between $C^{an}(\mathbb{Z}_p, K)^\vee$ (= loc. anal. distr. on $\mathbb{Z}_p$) and $\mathcal{R}_L^+$.

Take $\mu_\alpha \in B(\alpha)^\vee$ and define $\mu_{\alpha,i} \in C^{val(\alpha_p)}(\mathbb{Z}_p, K)^\vee \hookrightarrow C^{an}(\mathbb{Z}_p, K)^\vee$ as:

$$\langle \mu_{\alpha,i}, f(z)\rangle := \langle \mu_\alpha, \mathbb{1}_{\frac{1}{p^i}\mathbb{Z}_p} \cdot f(p^i z)\rangle$$

and let $w_{\alpha,i} \in \mathcal{R}_K^+$ such that $\alpha_p^{-i}\mu_{\alpha,i} \longmapsto w_{\alpha,i}$.

If $\mu_\alpha \in \left(B(\alpha)/L(\alpha)\right)^\vee$, then $\mu_\alpha \longmapsto \mu_\beta \in \left(B(\beta)/L(\beta)\right)^\vee \rightsquigarrow w_{\beta,i} \in \mathcal{R}_K^+$ analogously.

The map is:

$$\mu_\alpha \longmapsto \left(w_{\alpha,i}\otimes e_\alpha \oplus w_{\beta,i}\otimes e_\beta\right)_i \in \varprojlim_{\psi}\left(\mathcal{R}_K^+ \otimes_K D\right)$$

and one gets like this exactly the sequences such that:

(i) $\forall i \geq 0$, $\begin{cases} w_{\alpha,i} \text{ is of order} \leq val(\alpha_p) \\ w_{\beta,i} \text{ '' '' '' } \leq val(\beta_p) \end{cases}$ and the 2 sequences

$\left(\|w_{\alpha,i}\|_{val(\alpha_p)}\right)_{i\geq 0}$ and $\left(\|w_{\beta,i}\|_{val(\beta_p)}\right)_{i\geq 0}$ are bounded;

(ii) $\psi(w_{\alpha,i}) = \alpha_p^{-1} w_{\alpha,i-1}$ and $\psi(w_{\beta,i}) = \beta_p^{-1} w_{\beta,i-1}$;

(iii) $\forall j \in \{0, \dots, k-2\}$, $\forall i \geq 0$, $\forall m \geq m(V)$, $\forall \eta_{p^m}$ = primitive p^m-th root of 1 :

$$\underbrace{\left(\sum_{x \in \mathbb{Z}_p^\times / 1+p^{m(V)}\mathbb{Z}_p} (\beta\alpha^{-1})(x)\, \eta_{p^m}^{p^{m-m(V)}x} \right)}_{\text{Gauss sum}} \alpha_p^{m-i} \langle \mu_{\alpha,i}, \underbrace{z^j \eta_{p^m}^z}_{\substack{\text{loc. alg.} \\ \text{function on } \mathbb{Z}_p}} \rangle = \beta_p^{m-i} \langle \mu_{\beta,i}, z^j \eta_{p^m}^z \rangle$$

Here $m(V)$ is the smallest integer ≥ 1 such that $G(\beta\alpha^{-1}) \in (\mathbb{Q}_p(\zeta_{p^{m(V)}}) \otimes_{\mathbb{Q}_p} K)^\times$.

To conclude, one has to use the following theorem (due to Berger and Colmez) :

<u>Thm:</u> The space $(\varprojlim_\psi D(V))^b$ can be identified with the vector-subspace of $\varprojlim_\psi (R_K^+ \otimes_K D)$ of sequences $(w_{\alpha,i} \otimes e_\alpha + w_{\beta,i} \otimes e_\beta)_i$ satisfying (i), (ii) and (iii).

<u>rk:</u> Condition (iii) comes from a condition involving the Hodge filtration.