April 13. 2006. Thursday (2:30PM – 4:00PM)

Christophe Breuil – 2nd lecture

Aspect II) $(\varphi-\Gamma)$–modules of (Fontaine, Colmez.

Fix a compatible system of primitive p^n-roots of 1 (ζ_{p^n})

V = cts rep'n of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ / fin. dim K–v space.

$V \rightsquigarrow D(V) = (\varphi, \Gamma)$–module of V

$= \text{free } \left(K \underset{\mathcal{O}_K}{\otimes} \mathcal{O}_K[[X]][\frac{1}{X}]^{\wedge}\right)$–module of rk $=\dim_K V = n$

$$\varphi : D(V) \longrightarrow D(V)$$

injective semi-linear

$$\varphi(X \cdot v) = ((1+X)^p - 1) \cdot \varphi(v)$$

$$\Gamma := \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p) \underset{\varepsilon}{\cong} \mathbb{Z}_p^{\times}$$. ε = p-adic cyclotomic character.

$$\gamma : D(V) \longrightarrow D(V), \quad \gamma \in \Gamma$$

bijective semi-linear

$$\gamma(X \cdot v) = ((1+X)^{\varepsilon(\gamma)} - 1) \cdot \gamma(v)$$

$$\psi : D(V) \to D(V)$$

$v \in D(V)$, $v = \sum_{i=0}^{p-1} (1+X)^i \varphi(v_i)$. $v_i \in D(V)$

$$\psi(v) := v_0 \quad , \quad \psi \cdot \varphi = \mathrm{Id}.$$

"Weak topology" on $D(V) \cong \left(K \otimes \mathcal{O}_K[[X]][\frac{1}{X}]^{\wedge}\right)^n$

$$K \otimes \mathcal{O}_K[[X]][\tfrac{1}{X}]^\wedge = \bigcup_{n \geq 0} \frac{1}{\pi_K^n} \mathcal{O}_K[[X]][\tfrac{1}{X}]^\wedge$$

Funda. system of nbhd of 0

$$= \left(\pi_K^n \mathcal{O}_K[[X]][\tfrac{1}{X}]^\wedge + \frac{1}{X^m} \mathcal{O}_K[[X]] \right)_{\substack{n \geq 0 \\ m \leq 0}}$$

$(v_i)_i$ on $D(V)$

$(X_i)_i \in \mathcal{O}_K[[X]][\tfrac{1}{X}]^\wedge$

$(X_i)_i$ bdd if $\forall n \geq 0$, $\exists m \in \mathbb{Z}$ s.t. $X_i \in \pi_K^n \cdot \mathcal{O}_K[[X]] \cdot [\tfrac{1}{X}]^\wedge + \frac{1}{X^m} \mathcal{O}_K[[X]]$.

<u>The Basic recipe</u>

unitary Banach space

$+ \begin{pmatrix} 1 & \mathbb{Q}_p \\ 0 & \mathbb{Q}_p^\times \end{pmatrix} \times B(D) \longrightarrow B(D)$ continuous

$$\{(\varphi, \Gamma)\text{-modules}\} \longrightarrow \left\{ \begin{array}{l} \text{continuous unitary representations} \\ \text{of } \begin{pmatrix} 1 & \mathbb{Q}_p \\ 0 & \mathbb{Q}_p^\times \end{pmatrix} \text{ on } K\text{-Banach space} \end{array} \right\}$$

$$D \longmapsto B(D) = (\check{B}(D))^\vee = \mathrm{Hom}_{\mathcal{O}_K}^{\mathrm{conti}}(\check{B}(D^\circ), K), \quad D^\circ$$

D° = lattice in D

$$\check{B}(D) = \{ \text{bdd seq's in } \varprojlim_{\psi} D \}$$

not yet Banach space

$$= \{ (v_i)_{i \geq 0},\ v_i \in D,\ \psi(v_i) = v_{i-1} \}$$

In $\check{B}(D)$, take the induced topology by $\underbrace{\varprojlim_{\psi} D}_{\text{projective limit topology}}$.

$\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ acts as $(v_i)_i \longmapsto (\psi(v_i))_i$

$\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}_{a \in \mathbb{Z}_p^\times}$ acts as $(v_i)_i \longmapsto (\gamma_a v_i)$ $\quad \Gamma \xrightarrow[\varepsilon^{-1}]{\sim} \mathbb{Z}_p^\times$, $\gamma_a \longmapsto a$

$\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}_{z \in \mathbb{Z}_p}$ acts as $(v_i) \longmapsto (\varphi^i((1+X)^z) \cdot v_i)_i$

$f \in B(D)$. $\|f\| = \sup_{v^\vee \in \check{B}(D^\circ)} |f(v^\vee)|$ $\quad \check{B}(D^\circ) \simeq \varprojlim_{\psi} D^{\circ,\#}$, $D^{\circ,\#}$: cpt. (Laurent Berger's lecture) local "p-adic Langlands"

$$\left\{\begin{array}{l}\text{conti rep'n of } Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)\\ \text{on } K\text{-v. space } V\end{array}\right\} \longrightarrow \{(\varphi,\Gamma)\text{-module}\}$$

$$\begin{tikzcd} V \arrow[r, mapsto] \arrow[dr] & D(V) \arrow[d] \\ & B(D(V)) = B(V) \end{tikzcd}$$

Lemma: If V is abs. irreducible of dim >1, then
$B(V)$ is topol. irreducible as a rep'n of $\begin{pmatrix} 1 & \mathbb{Q}_p \\ 0 & \mathbb{Q}_p^\times \end{pmatrix}$

Sketch of Proof) : prove $B(V)^\vee$ is top. irreducible.
(Colmez)

Assume V is abs. irred of dim >1, then there is a unique non-zero K-v. subspace $D^\#(V) \subset D(V)$ containing an $\mathcal{O}_K[[X]]$-submodule $D^{\#,0}$ that is bdd, preserved by ψ with ψ surjective and generating over K

i.e. $D^{\#,0}/p^n$ is of finite type over $\mathcal{O}_K/p^n[[X]]$ $\forall n$.

$T \subset V$ $\quad B^\vee(D)$

$$0 \neq B \subseteq \left(\varprojlim_{\psi} D(V)\right)^b = \left(\varprojlim_{\psi} D(T)\right)^b \otimes K$$

$\hookrightarrow$ closed, preserved by $\begin{pmatrix} 1 & \mathbb{Q}_p \\ 0 & \mathbb{Q}_p^\times \end{pmatrix}$

prove $B = B^\vee(V)$

prove $B = B^\vee(V)$.

$$B \simeq \left(\varprojlim_{\psi} M^0\right)^b \otimes K$$

$$M^0 := \left\{ v \in D(T) \;\middle|\; \exists (v_i) \in B \cap \left(\varprojlim_{\psi} D(T)\right)^b \text{ s.t. } v_0 = v \right\}$$

easy to check, using the fact B is preserved by $\begin{pmatrix} 1 & \mathbb{Q}_p \\ 0 & \mathbb{Q}_p^\times \end{pmatrix}$

$\psi : M^\circ \to M^\circ$, is surjective and $B \simeq \left(\varprojlim_{\psi} M^\circ\right)^b \otimes K$

Exercise

$B \neq 0 \Rightarrow M^\circ \neq 0$.

$M^\circ \cong D^\#(T) \Rightarrow M^\circ$ compact

$\Rightarrow M^\circ$ bdd as an $\mathcal{O}_K[[X]]$-module

$\psi : M^\circ \twoheadrightarrow M^\circ$

$M^\circ \otimes K = D^\#(V)$. $\quad \left(\varprojlim_{\psi} D^\#(V)\right)^b \overset{\text{Laurent Berger's Talk}}{=} \left(\varprojlim_{\psi} D(V)\right)^b$

$$\begin{array}{c} \| \\ \left(\varprojlim_{\psi} D^\#(T)\right)^b \otimes K \end{array}$$

$$\simeq \left(\varprojlim_{\psi} M^\circ \otimes K\right) \simeq B$$

<u>Hope 1</u> : If V is <u>abs. irred.</u> of dim 2, then the action on $B(V)$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ missing — extends uniquely to $GL_2(\mathbb{Q}_p)$.

(Colmez – using explicit reciprocity law)

<u>Hope 2</u> : If V is moreover de Rham, with $\neq$ HT weights, then $B(V)$ (with extended action) is a unitary completion of $\rho(V) \underset{K}{\otimes} \Pi(V)$ where $\rho(V) = \mathrm{Sym}^{i_2 - i_1 - 1} K^2 \otimes \det^{i_1}$

$HT(V) = i_1 < i_2$

$$\left(\varphi, N, \mathrm{Gal}(L/\mathbb{Q}_p), D\right) \xmapsto{WD + F_{ss}} (r, N, V) \longleftrightarrow \Pi(V)$$

see previous talk.

<u>The case</u> $\Pi(V)$ = principal series (and V abs. irred of dim 2)

The most simple case $(r, \overset{0}{\overset{\|}{N}})$ completely determines V.

<u>Thm</u> (Berger–B) Assume that D is φ-semi stable, then one can extend the action of $\begin{pmatrix} 1 & \mathbb{Q}_p \\ 0 & \mathbb{Q}_p^\times \end{pmatrix}$ on $B(V)$ to an action of $GL_2(\mathbb{Q}_p)$ such that $B(V)$ ...

Moreover, there is <u>no</u> <u>other</u> <u>non-zero</u> unitary <u>completion</u> of $\rho(V)\otimes\pi(V)$ in that case

<u>Steps of proof</u>

$k\geq 2 \quad \alpha,\beta : \mathbb{Q}_p^\times \longrightarrow K^\times$ smooth characters

$0<\mathrm{val}(\alpha_p)<k-1 \qquad \alpha_p := \alpha(p)^{-1}$ <u>distinct</u>

$0<\mathrm{val}(\beta_p)<k-1 \qquad \beta_p := \beta(p)^{-1}$ <u>trivial</u> on $1+p^n\mathbb{Z}_p$, $n\geq 1$

$\mathrm{val}(\alpha_p)+\mathrm{val}(\beta_p)=k-1$

$$D = K\cdot e_\alpha \oplus K\cdot e_\beta \qquad \begin{cases}\varphi(e_\alpha)=\alpha_p^{-1}\cdot e_\alpha\\ \varphi(e_\beta)=\beta_p^{-1}\cdot e_\beta\end{cases}$$

$$\begin{cases}\gamma(e_\alpha) = \alpha(\varepsilon(\gamma))\cdot e_\alpha\\ \gamma(e_\beta) = \beta(\varepsilon(\gamma))\cdot e_\beta\end{cases} \qquad \gamma\in\Gamma$$

$$\mathrm{Fil}^i(\mathbb{Q}_p(\zeta_{p^n})\otimes_{\mathbb{Q}_p} D) = \begin{cases}\text{all} & \text{if } i\leq -(k-1)\\ \mathbb{Q}_p(\zeta_{p^n})\otimes_{\mathbb{Q}_p} K\cdot(e_\alpha + G(\beta\alpha^{-1})e_\beta) & \text{if } -(k-1)\leq i\leq 0\\ 0 & \text{if } i\geq 1\end{cases}$$

Gauss sum

$$G(\beta\alpha^{-1}) = 1 \quad \text{if } \beta\alpha^{-1} \text{ is } \underline{\text{unramified}}.$$

$$G(\beta\cdot\alpha^{-1}) \in (\mathbb{Q}_p(\zeta_{p^n})\otimes_{\mathbb{Q}_p} K)^\times$$

alg smooth

$\rho\otimes\pi$

irred or alg. rep^n

$$= \sum_{\gamma\in\Gamma/\Gamma_n} \gamma(\zeta_{p^n})\otimes(\beta\cdot\alpha^{-1})^{-1}(\gamma)$$

$$\Gamma_n = 1+p^n\mathbb{Z}_p$$

$$\rho\otimes_K\pi = \mathrm{Sym}^{k-2}K^2\otimes_K \mathrm{Ind}_{\begin{pmatrix}* & *\\ 0 & *\end{pmatrix}}^{GL_2(\mathbb{Q}_p)} \alpha\otimes\beta\cdot|\cdot|^{-1}$$

<u>Π : irred</u> $\quad GL_2(\mathbb{Q}_p)$ on $B(V) \rightsquigarrow (\varphi,\Gamma)$-module

<u>Step 1.</u> : describe the completion of $\rho\otimes_K\pi$ w.r.t any generating $\mathcal{O}_K[GL_2(\mathbb{Q}_p)]$-submodule of fin. type.

<u>Step 2</u> : Identify this with $B(V)$

<u>Step 1</u> : Compute the dual of this completion.

Fix. some $\mathcal{O}_K[GL_2(\mathbb{Q}_p)]$-submodule of fin. type in $\rho \otimes \pi$, M.

and compute which linear forms $\mu : \rho \otimes_K \pi \to K$ satisfies $\|\mu(m)\| \leq 1$, $\forall m \in M$.

writing $\mathrm{Sym}^{k-2} K^2 \simeq \bigoplus_{j=0}^{k-2} K \cdot z^j$.

$\rightsquigarrow$ $\forall a \in \mathbb{Q}_p$. $\forall j \in \{0, \cdots, k-2\}$. $\forall n \in \mathbb{Z}$.

(i) $\displaystyle\int_{a+p^n\mathbb{Z}_p} (z-a)^j \, d\mu(z) \in C_M \cdot p^{n(j - \mathrm{val}(\alpha_p))} \mathcal{O}_K$) μ is tempered of order $\leq \mathrm{val}(\alpha_p)$

(ii) $\displaystyle\int_{\mathbb{Q}_p - (a+p^n\mathbb{Z}_p)} \beta \cdot \alpha^{-1} (z-a) \cdot |z-a|^{-1} \cdot (z-a)^{k-2-j} \, d\mu(z) \in C_M \cdot p^{n(\mathrm{val}(\alpha_p) - j)} \mathcal{O}_K$

$\sum_{n=0}^{\infty} a_n \binom{z}{n}$ | $n^r |a_n| \to 0$, $n \to \infty$ | μ is tempered of order $\leq \mathrm{val}(\alpha_p)$

$$(\rho \otimes_K \pi)^\wedge = B(\alpha) / L(\alpha)$$

$L(\alpha)$ → closure of subgp gen. by $z \mapsto z^j$, $z \mapsto \beta\alpha^{-1}(z-a)|z-a|^{-1}(z-a)^{k-2-j}$

<u>Step</u> 2:

$$\left(B(\alpha)/L(\alpha)\right)^{\vee} \xrightarrow{\sim} \left(\varprojlim_{\psi} D(V)\right)^b$$

dual

Laurent's thesis (D_{cris} is needed)

Amice transform.

$\mathcal{R}_K^+ = \left\{ \sum_{n=0}^{\infty} a_n X^n \;\middle|\; \text{converging on the open unit disc} \right\}$

$\cap$ $K[[X]]$

$\mathcal{R}_K^+ \leftarrow C^{an}(\mathbb{Z}_p, K)^{\vee}$ Amice transform

$\sum_{n=0}^{\infty} \mu\left(\binom{z}{n}\right) \cdot X^n \longleftarrow \mu$ (p-adic F. T)

$\mu_\alpha \in B(\alpha)^{\vee} \to \mu_{\alpha,i} \in C^{\mathrm{val}(\alpha_p)}(\mathbb{Z}_p, K)^{\vee} \hookrightarrow C^{an}(\mathbb{Z}_p, K)^{\vee}$

$$\langle \mu_{\alpha,i}, f \rangle := \langle \mu_\alpha, \mathbb{1}_{\frac{1}{p^i}\mathbb{Z}_p} f(p^i z) \rangle$$

$w_{\alpha,i} \longleftrightarrow \alpha_p^{-1} \cdot \mu_{\alpha,i}$

$\cap$ $\mathcal{R}_K^+$

If $\mu_\alpha \in \left(B(\alpha)/L(\alpha)\right)^\vee \underset{\nearrow}{\cong} \left(B(\beta)/L(\beta)\right)^\vee$

Intertwining $\cup$ same way

$\mu_\beta \rightsquigarrow (w_{\beta_i})_i$

$$\mu_\alpha \longmapsto \left(w_{\alpha,i} \otimes e_\alpha + w_{\beta,i} \otimes e_\beta\right)_i$$

$$\in \varprojlim_{\psi} \left(R_K^+ \otimes_K D\right)$$

easy check

Filtered module

$\psi := \varphi^{-1}$ on D

get like this exactly

the sequence of such elements

such that (i) $\forall i \geq 0$, $w_{\alpha,i}$ is of order $\leq val(\alpha_p)$ tempered dist

$w_{\beta,i}$ ——— $\leq val(\beta_p)$.

and the seq $\left(\|w_{\alpha,i}\|_{val(\alpha_p)}\right)_{i\geq 0}$ are bdd

$\left(\|w_{\beta,i}\|_{val(\beta_p)}\right)_{i\geq 0}$

(ii)

$$\psi(w_{\alpha,i}) = \alpha_p^{-1} \cdot w_{\alpha,i-1}$$

$$\psi(w_{\beta,i}) = \beta_p^{-1} \cdot w_{\beta,i-1}$$

$m(V) \geq 1$ smallest integer such that $G(\beta \cdot \alpha^{-1}) \in (\mathbb{Q}_p(\zeta_{p^{m(V)}}) \otimes K)^\times$.

(iii) $\forall j \in \{0, 1, \ldots, k-2\}$, $\forall i \geq 0$, $\forall m \geq m(V)$

$\forall \eta_{p^m}$ = prim. p^m- root of 1.

$$\left(\sum_{x \in \mathbb{Z}_p^\times/1+p^{m(V)}\mathbb{Z}_p} (\beta \cdot \alpha^{-1})(x) \cdot \eta_{p^m}^{p^{(m-m(V))}x}\right) \alpha_p^{m-i} \langle \mu_{\alpha,i} \, , \, z^j \cdot \eta_{p^m}^z \rangle$$

locally alg. ftn on $\mathbb{Z}_p$

$$= \beta_p^{m-i} \langle \mu_{\beta,i} \, , \, z^j \cdot \eta_{p^m}^z \rangle$$

<u>Thm</u> (L. Berger)

The sp. $\left(\varprojlim_{\psi} D(V)\right)^b$ can be identified with the vector subspace of $\varprojlim_{\psi} (R_K^+ \otimes D)$ of sequences $(w_{\alpha,i} \otimes e_\alpha + w_{\beta,i} \otimes e_\beta)_{i \geq 0}$

satisfying (i), (ii) $\underbrace{\text{(iii)}}$.

condition involving the Hodge filtration