Apr. 10th Galois Representations & (φ,Γ)-Modules (1)

by Laurrent Berger

① The $\mathbb{F}_p$-reps of $G = Gal(\mathbb{F}_p((X))^{sep}/\mathbb{F}_p((X)))$

$E = \mathbb{F}_p((X))^{sep}$ $\varphi : y \mapsto y^p$

$V = \mathbb{F}_p$-rep of G

$\exists\, k/\mathbb{F}_p((X))$ finite, s.t. $V|G_k$ is trivial by Hilbert 90

we have $k \otimes_{\mathbb{F}_p} V \simeq k^d$ as semi-linear reps of G

We get $D(V) = (E \otimes_{\mathbb{F}_p} V)^G$, it is a $\mathbb{F}_p((X))$-vector space of dim d

with a Frob. map $\varphi : D(V) \to D(V)$ which is injective (so that $\varphi^* D(V) = D(V)$)

$D(V)$ is a φ-module over $\mathbb{F}_p((X))$

We have : $E \otimes_{\mathbb{F}_p((X))} D(V) \simeq E \otimes_{\mathbb{F}_p} V$ so that $V = (E \otimes_{\mathbb{F}_p((X))} D(V))^{\varphi=1}$

<u>Propn</u> The functor $V \mapsto D(V)$

$\{\mathbb{F}_p\text{-reps of } G\} \longrightarrow \{\varphi\text{-modules over } \mathbb{F}_p((X))\}$

is an equivalence of categories

② Choose $0 < s \leq 1$, and let $I = \{x \in \mathcal{O}_{\mathbb{C}_p}, val(x) \geq s\}$

Let $\widetilde{\mathbb{E}}^+ = \{(x_0, x_1, \ldots)\ x_i \in \mathcal{O}_{\mathbb{C}_p}/I,\ x_{i+1}^p = x_i\}$

$= \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_p}/I$

$\widetilde{\mathbb{E}}^+$ is a ring of char. p $\alpha \in \overline{\mathbb{F}}_p$, $([\alpha], [\alpha^{1/p}], \ldots) \in \widetilde{\mathbb{E}}^+ \Rightarrow \overline{\mathbb{F}}_p \subset \widetilde{\mathbb{E}}^+$

Define a valuation v_E on $\widetilde{\mathbb{E}}^+$ by $v_E(x) = \lim_{i \to +\infty} p^i val(\widetilde{x}_i)$ $\widetilde{x}_i \in \mathcal{O}_{\mathbb{C}_p}$ lifts x_i

· Choose for $n \geq 0$, $\varepsilon^{(n)}$ a p^n-th root of unity, s.t. $(\varepsilon^{(n+1)})^p = \varepsilon^{(n)}$, $\varepsilon^{(1)} \neq 1$

$\varepsilon = (1, \varepsilon^{(1)}, \varepsilon^{(2)}, \ldots) \in \widetilde{\mathbb{E}}^+$

$\pi = \varepsilon - 1$, $v_E(\pi) = p/p-1$

We define $\widetilde{\mathbb{E}} = \widetilde{\mathbb{E}}^+[\frac{1}{X}]$

rop. (1) $\widetilde{\mathbb{E}}$ is a complete ~~valuation~~ valued field, whose ring of integers is $\widetilde{\mathbb{E}}^+$
(2) $\widetilde{\mathbb{E}}$ is algebraically closed.
(3) $\mathbb{F}_p((\pi)) \subset \widetilde{\mathbb{E}}$ and $\mathbb{E} = \mathbb{F}_p((X))^{sep}$ is dense in $\widetilde{\mathbb{E}}$

We see that $G_{\mathbb{Q}_p} = Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ acts on $\mathbb{E} \subset \widetilde{\mathbb{E}}$

et $F_n = \mathbb{Q}_p(\mu_{p^n})$, $F_\infty = \bigcup_{n\geq 0} F_n$, $\Gamma = Gal(F_\infty/\mathbb{Q}_p) \overset{\chi}{\simeq} \mathbb{Z}_p^\times$, χ=cyclo. character

$g(\varepsilon^{(n)}) = (\varepsilon^{(n)})^{\chi(g)} \Rightarrow \cancel{g(X)}\, g(\pi) = (1+\pi)^{\chi(g)} - 1$

If $h \in H_{\mathbb{Q}_p} = \ker \chi = Gal(\overline{\mathbb{Q}}_p/F_\infty)$, then h acts trivially on $\mathbb{F}_p((\pi))$

We get a map $H_{\mathbb{Q}_p} \longrightarrow G_{\mathbb{F}_p((\pi))}$, which is an isomorphism!!

The "field of norms" of Fontaine & Wintenberger

K= finite ext^n of $\mathbb{Q}_p$, $K_n = K(\mu_{p^n})$, $K_\infty = \bigcup_{n\geq 0} K_n$

$\mathcal{N}_{K_\infty/K} = \{(x^{(0)}, x^{(1)}, \cdots) \mid x^{(i)} \in K_i,\ N_{K_{i+1}/K_i}(x^{(i+1)}) = x^{(i)}\}$

By higher ramification theory, $N_{K_{i+1}/K_i} \approx (x \mapsto x^p)$ so that we have a map: $\iota : \varprojlim_N \mathcal{O}_{K_i} \hookrightarrow \varprojlim \mathcal{O}_{\mathbb{C}_p}/I$

$$x = (x^{(i)}) \longmapsto \iota(x) = (x_i)$$, where $x_i = (x^{(i+j)})^{p^j}$ for $j \gg 0$

which extends to $\iota : \mathcal{N}_{K_\infty/K} \hookrightarrow \widetilde{\mathbb{E}}$

rop. (1) $\iota(\mathcal{N}_{F_\infty/F}) = \mathbb{F}_p((\pi))$
(2) $\iota(\mathcal{N}_{K_\infty/K})$ is a finite separable ext^n of $\mathbb{F}_p((\pi))$ equal to $\mathbb{E}^{H_K}$
(3) One gets every finite separable ext^n of $\mathbb{F}_p((\pi))$ in this way, and $H_{\mathbb{Q}_p} \simeq G_{\mathbb{F}_p((\pi))}$

xample: If $K/\mathbb{Q}_p$ is unramified, then $\iota(\mathcal{N}_{K_\infty/K}) \simeq k_K((\pi))$ and we can see this explicitly.

If $y \in \varprojlim_N \mathcal{O}_{K_i}$, then Coleman proved $\exists!\ \mathrm{Col}_y(X) \in \mathcal{O}_K[\![X]\!]$ such that $\mathrm{Col}_y^{\sigma^{-n}}(\varepsilon^{(n)}-1) = y^{(n)}$ $\forall n \geq 1$
and then $\iota(y) = \overline{\mathrm{Col}_y} \in k_K[\![X]\!] = k_K[\![\pi]\!]$

-) (φ,Γ)-modules ~~in char~~ for $\mathbb{F}_p$-reps of $G_{\mathbb{Q}_p}$

If V is an $\mathbb{F}_p$-rep of $G_{\mathbb{Q}_p}$, $D(V) = (\mathbb{E} \otimes_{\mathbb{F}_p} V)^{H_{\mathbb{Q}_p}}$ is a φ-module over $\mathbb{F}_p((\pi))$ and $\Gamma = G_{\mathbb{Q}_p}/H_{\mathbb{Q}_p}$ acts on $D(V)$
and $D(V)$ is a (φ,Γ)-module over $\mathbb{F}_p((\pi))$
One can recover V by $V = (\mathbb{E} \otimes_{\mathbb{F}_p((\pi))} D(V))^{\varphi=1}$ $\quad g(e\otimes y) = g(e) \otimes \bar{g}(y)$
This gives rise to an equivalence of category
$\{\mathbb{F}_p\text{-reps of } G_{\mathbb{Q}_p}\} \longrightarrow \{(\varphi,\Gamma)\text{-modules over } \mathbb{F}_p((\pi))\}$

This works equally well in char 0
Let $\mathcal{E} = \{\sum_{i\in\mathbb{Z}} a_i \pi^i,\ a_i \in \mathbb{Q}_p,\ \{a_i\}_{i\in\mathbb{Z}} \text{ bounded and } a_i \to 0, \text{ as } i \to -\infty\}$
$\mathcal{O}_\mathcal{E} = \{f(x) \in \mathcal{E},\ a_i \in \mathbb{Z}_p,\ \forall i\}$
so that $\mathcal{E} = \mathcal{O}_\mathcal{E}[\frac{1}{p}]$ & $\mathcal{O}_\mathcal{E}/p = \mathbb{F}_p((\pi))$
Define φ and action of Γ: $\varphi(f(x)) = f((1+x)^p - 1)$
$\gamma(f(x)) = f((1+x)^{\chi(\gamma)} - 1)$

Def^n A (φ,Γ)-module over $\mathcal{O}_\mathcal{E}$ (or $\mathcal{E}$) is an $\mathcal{O}_\mathcal{E}$ (or $\mathcal{E}$)-module of finite type, with semi-linear, commuting φ and (continuous)* action of Γ.
*weak topology
Over $\mathcal{O}_\mathcal{E}$, we require $\varphi^*(D) = D$
Over $\mathcal{E}$, we say φ is étale if it has an $\mathcal{O}_\mathcal{E}$-lattice D^0 stable under φ s.t. $\varphi^*(D_0) = D_0$

- $\{\mathbb{Z}_p\text{-reps of } G_{\mathbb{Q}_p}\} \longleftrightarrow \{(\varphi,\Gamma)\text{-modules over } \mathcal{O}_\mathcal{E}\}$
$\{\mathbb{Q}_p\text{-reps of } G_{\mathbb{Q}_p}\} \longleftrightarrow \{\text{étale } (\varphi,\Gamma)\text{-modules over } \mathcal{E}\}$