Spectral questions in endoscopic transfer for real reductive groups

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Introduction

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- second principle, **stable transfer**, concerns stable harmonic analysis on any two groups $G(\mathbb{R})$, $H(\mathbb{R})$ related by a morphism of L-groups, part of *Beyond Endoscopy*, not discussed here

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- matching: based on norm correspondence for very regular stable conjugacy classes in $H_1(\mathbb{R})$ and (twisted) classes in $G(\mathbb{R})$
- matching provides a transfer of test functions from $G(\mathbb{R})$ to $H_1(\mathbb{R})$, then a dual map from $\mathfrak{Z}$-finite stable distributions on $H_1(\mathbb{R})$ to $\mathfrak{Z}$-finite invariant distributions on $G(\mathbb{R})$

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- **spectral transfer:** interpret this dual map in terms of traces of irreducible admissible representations

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- apply this to various known identities to get (partial) spectral transfer
- the spectral factors contain precise information needed about packets

Endoscopic transfer: geometric side

a. general twisted setting

- G connected, reductive algebraic group defined over $\mathbb{R}$
  θ an $\mathbb{R}$-automorphism of G, ϖ a quasi-character on $G(\mathbb{R})$
  study representations π for which $\pi \circ \theta \simeq \varpi \otimes \pi$

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- **quasi-split data** (G^*, θ^*) **:**
  G^* quasi-split over $\mathbb{R}$, has an $\mathbb{R}$-splitting $spl^* = (B^*, T^*, \{X_\alpha\})$
  [ultimately choice of spl^* will not matter]
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- **inner form** (G, θ, η) **of** (G^*, θ^*) **:**
  (G, θ) as above, and $\eta : G \to G^*$ an inner twist such that
  η transports θ to θ^* up to inner automorphism:
  $\theta = Int(h_\theta) \circ \eta^{-1} \circ \theta^* \circ \eta$, where $h_\theta \in G$

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- up to **isomorphism** of inner forms, can arrange that transport
  $\eta^{-1} \circ \theta^* \circ \eta$ is defined over $\mathbb{R}$, so $Int(h_\theta) \in G_{ad}(\mathbb{R})$
  [use **fundamental splittings** – exist for all G]

- **dual complex group** $G^\vee$ with splitting $spl^\vee$ dual to spl^*, action of Weil group $W_\mathbb{R}$ through $W_\mathbb{R} \to \Gamma = Gal(\mathbb{C}/\mathbb{R}) = \{1, \sigma\}$ action preserves $spl^\vee$, **and** L**-group** ${}^L G = G^\vee \rtimes W_\mathbb{R}$

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b. dual data

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- in talk: assume $G^\vee$-component of a is **bounded**, so ϖ unitary [otherwise, insert *essentially* in various statements ...]

- **(bounded) supplemented endoscopic data $\mathfrak{e}_z$:**
  endoscopic data $\mathfrak{e} = (H, \mathcal{H}, s)$, together with
  z-extension data (H_1, ξ_1) [Kaletha refinement ...]

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- **basic picture:**

$$
\begin{array}{ccccccccc}
 & & & & & & {}^L H_1 & & \\
 & & & & & \overset{\xi_1}{\nearrow} & & & \\
1 & \longrightarrow & Cent_{\theta^\vee}(s, G^\vee)^0 & \longrightarrow & \mathcal{H} & \rightleftarrows & W_{\mathbb{R}} & \longrightarrow & 1 \\
 & & & & & \underset{incl}{\searrow} & & & \\
 & & & & & & {}^L G & &
\end{array} \qquad (1)
$$

where $W_{\mathbb{R}}$ acts on $Cent_{\theta^\vee}(s, G^\vee)^0 = H^\vee$ by conjugation by elements of $Cent_{^L\theta_a}(s, {}^L G)$

- in talk: assume θ **preserves a fundamental splitting**
  [at each step should note effect of extra twist by elt of $G_{ad}(\mathbb{R})$]

Endoscopic transfer: geometric side

c. norm correspondence

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- there is Γ-map $\mathcal{A}$ from the set $Cl_{ss}(H_1)$ of semisimple conjugacy classes in $H_1(\mathbb{C})$ to the set $Cl_{\theta\text{-}ss}(G,\theta)$ of θ-semisimple θ-conjugacy classes in $G(\mathbb{C})$:

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- γ_1 is **strongly** G**-regular** if and only if $\mathcal{A}$ maps its class to a class of strongly θ-regular elements in G
- strongly G-regular γ_1 **is a norm of** strongly θ-regular δ, *i.e.* (γ_1,δ) is a **norm pair**, if and only if δ is in image of class of γ_1

- sufficient to specify geometric transfer on **very regular set**: all pairs $(\gamma_1, \delta) \in H_1(\mathbb{R}) \times G(\mathbb{R})$, where γ_1 is strongly G-regular and δ is strongly θ-regular

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- two versions of transfer: here use factors for classical version other version: (turns out to be) complex conjugate

- definitions allow simultaneously treatment of inner forms
  extended group = K-group: fills out stable conjugacy classes

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- Δ_I, Δ_{III} have Galois-cohomological definitions, spectral versions in same groups [sample at end of talk]

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  spectral versions in same groups [sample at end of talk]

- $\Delta_{II}(\gamma_1, \delta)$ comes from analysis of jumps in orbital integrals
  spectral version: different form, involves character formula

Endoscopic transfer: geometric side

ddd. transfer factors (cont.)

- **toral data** associated with norm pair (γ_1, δ) : there is θ^*-stable pair (B, T) in G^*, with T defined over $\mathbb{R}$, and various maps yielding

$$\begin{array}{ccccccc} & & \delta & \overset{inner}{\rightsquigarrow} & \delta^* \in T(\mathbb{C}) \\ & & & & \downarrow \\ \gamma_1 & \overset{z}{\longrightarrow} & \gamma_H & \overset{endo}{\longleftrightarrow} & \gamma^* \in T_{\theta^*}(\mathbb{R}) \end{array} \tag{4}$$

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- $R_{res} = \theta^*$-restricted root system for T in G^*,Galois orbits $\mathcal{O}_{res}$
  $R_1 =$ root system for T_1 in H_1,Galois orbits $\mathcal{O}_1$
  to each indivisible $\mathcal{O}_{res}$ attach well-defined $\chi_\alpha(\frac{N\alpha(\delta^*)^{r_\alpha} - 1}{a_\alpha})$
  to each $\mathcal{O}_1$ attach well-defined $\chi_{\alpha_1}(\frac{\alpha_1(\gamma_1)-1}{a_{\alpha_1}})$ [notation]
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- χ-**data**, a-**data:** $\{\chi_\alpha\}, \{a_\alpha\}$ etc. as above
- same data used in Δ_I, Δ_{III} ; two of the three affect each of relative $\Delta_I, \Delta_{II}, \Delta_{III}$ but product Δ is independent of all choices

Theorem

For each θ-Schwartz fdg on $G(\mathbb{R})$ there exists λ_1-Schwartz $f_1 dh_1$ on $H_1(\mathbb{R})$ such that

$$SO(\gamma_1, f_1 dh_1) = \sum_{\delta,\ \theta\text{-conj}} \Delta(\gamma_1, \delta)\ O^{\theta,\varpi}(\delta, fdg) \tag{5}$$

for all strongly G-regular γ_1 in $H_1(\mathbb{R})$.

Corollary

If f has compact support then we may take f_1 of compact support mod $Z_1(\mathbb{R})$.

- corollary follows immediately from a theorem of Bouaziz

Endoscopic transfer: geometric side

f. remarks on statement

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- notation: $Z_1 = Ker(H_1 \longrightarrow H)$, $\mathfrak{e}_z$ determines character λ_1 on $Z_1(\mathbb{R})$, require $f_1(z_1 h_1) = \lambda_1(z_1)^{-1} f_1(h_1)$ for $z_1 \in Z_1(\mathbb{R})$, $h_1 \in H_1(\mathbb{R})$

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- (θ, ϖ)-twisted orbital integral

$$O^{\theta,\varpi}(\delta, fdg) := \int_{Cent_\theta(\delta,G)(\mathbb{R})\backslash G(\mathbb{R})} f(g^{-1}\delta\theta(g))\varpi(g)\frac{dg}{dt_\delta} \quad (6)$$

- corollary follows immediately from a theorem of Bouaziz
- notation: $Z_1 = Ker(H_1 \rightarrow H)$, $\mathfrak{e}_z$ determines character λ_1 on $Z_1(\mathbb{R})$, require $f_1(z_1 h_1) = \lambda_1(z_1)^{-1} f_1(h_1)$ for $z_1 \in Z_1(\mathbb{R})$, $h_1 \in H_1(\mathbb{R})$
- $\Delta(\gamma_1, \delta)$ is invariant under stable conjugacy in first variable, also has correct behavior under translation by $Z_1(\mathbb{R})$
- $SO(\gamma_1, f_1 dh_1)$ is usual normalized stable orbital integral
- left and right: compatible Haar measures in denominators of quotients
- (θ, ϖ)-twisted orbital integral

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- $\Delta(\gamma_1, \delta)$ has correct behavior under θ-conjugacy to make right side of (5) well-defined

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Endoscopic transfer: geometric side

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- (long) calculations with transfer factors to check these properties ■

- for each test fdg on $G(\mathbb{R})$ attach test $f_1 dh_1$ on $H_1(\mathbb{R})$ with matching orbital integrals in the sense of (5) of main theorem

Endoscopic transfer: spectral side

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- formula for Θ_1 as smooth function on regular set $\Longrightarrow$ formula for Θ as smooth function on regular set

- **goal:** for a stable character $\Theta_1 = St\text{-}Trace\ \pi_1$, where π_1 irreducible admissible representation of $H_1(\mathbb{R})$ with correct $Z_1(\mathbb{R})$ behavior, **to describe** Θ explicitly as a combination of (θ, ϖ)-twisted traces

$$f \longrightarrow Trace\ [\pi(f) \circ \pi(\theta, \varpi)] \tag{7}$$

notation: $\pi(\theta, \varpi)$ intertwines $\pi \circ \theta$ and $\varpi \otimes \pi$ [also drop dg, dh]

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- term on right side will be independent of normalization of $\pi(\theta, \varpi)$ [Δ_{II} involves twisted character formula and effects cancel]

Endoscopic transfer: spectral side

bb. dual transfer as spectral transfer (cont.)

- in place of very regular norm pairs $(\gamma_1, \delta), (\gamma_1', \delta')$, consider very regular related pairs $(\pi_1, \pi), (\pi_1', \pi')$: define (almost) canonical $\Delta(\pi_1, \pi; \pi_1', \pi')$

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- **standard setting:** $\theta =$ *identity*, $\varpi =$ *trivial character*
  results $\Longrightarrow$ structure on packets of representations
  ... then twisted setting $\Longrightarrow$ compatible additional structure
  on packets preserved by $\pi \to \varpi^{-1} \otimes (\pi \circ \theta)$

Endoscopic transfer: spectral side

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- define group $\mathcal{M} = \mathcal{M}_\varphi$ in ${}^L G$ as subgp gen by $M^\vee$ and $\varphi(W_\mathbb{R})$
$$1 \longrightarrow M^\vee \longrightarrow \mathcal{M} \rightleftarrows W_\mathbb{R} \longrightarrow 1$$
extract L-action same way as endo, $M^* =$ dual, quasi-split over $\mathbb{R}$

Endoscopic transfer: spectral side

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- attach packet Π to u-regular ψ : L-packet if $\rho = triv$, or Arthur packet otherwise [see Adams-Johnson, just elliptic here]

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- (ψ_1, ψ) **very regular pair**: ψ_1, ψ are u-regular and ψ_1 is G-regular
- very regular **related** pair: also $\psi = \psi_{\psi_1}$

Endoscopic transfer: spectral side

d. standard setting: tempered pairs

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  (8) says: $\mathit{St\text{-}Trace}\ \pi_1(f_1) = \sum_{\pi} \Delta(\pi_1, \pi)\ \mathit{Trace}\ \pi(f)$
  (π_1, π), (π_1', π') related pairs discrete series representations with Langlands parameters (φ_1, φ), (φ_1', φ')

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- Δ_{II} involves local formula for $Trace\ \pi(f)$ as smooth function ... [fourth root of unity if rewrite usual Harish-Chandra formula]

- via parabolic induction extend defns to $\Delta(\pi_1, \pi; \pi_1', \pi')$, $\Delta(\pi_1, \pi; \gamma_1, \delta)$, for all very regular norm pairs (γ_1, δ) and all tempered very regular related pairs $(\pi_1, \pi), (\pi_1', \pi')$ [set $\Delta(\pi_1, \pi) = 0$ if pair not related]

Endoscopic transfer: spectral side

dd. standard setting: tempered pairs (cont.)

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- **proof of (8) for tempered very regular pairs:** reduce quickly to elliptic case, discrete series both sides, and then apply Harish-Chandra characterization theorem: transfer Θ is tempered invariant eigendistribution with correct infinitesimal character and agrees with $\sum_{\pi} \Delta(\pi_1, \pi)$ *Trace* $\pi(f)$ on regular elliptic set

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- now **theorem for all tempered pairs?** for example, need this for converse: spec transfer for $(f_1, f) \Longrightarrow$ geom transfer for (f_1, f)

- main case = elliptic on left: transfer discrete series to limits of discrete series, limits which arise have Levi $\mathcal{M}$ of type $(A_1)^n$ then Hecht-Schmid character identities + analysis in $G^\vee$ identifies transfer Θ as right side of (8), where factor $\Delta(\pi_1, \pi)$ is defined via analog of Zuckerman translation for parameters

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- conclude the following continuation of geom transfer thm, std setting:

Theorem

Suppose geom, spec factors Δ are compatible. Then

$$St\text{-}Trace\ \pi_1(f_1 dh_1) = \sum_{\pi} \Delta(\pi_1, \pi)\ Trace\ \pi(fdg) \tag{9}$$

for all tempered irreducible admissible representations π_1 such that $Z_1(\mathbb{R})$ acts by λ_1.

- **Conversely:** *if* fdg, $f_1 dh_1$ *are test measures satisfying (9) then*

$$SO(\gamma_1, f_1 dh_1) = \sum_{\delta\ conj} \Delta(\gamma_1, \delta)\ O(\delta, fdg) \tag{10}$$

for all strongly G*-regular* γ_1 *in* $Z_1(\mathbb{R})$.
Proof: Use both transfer thms plus *same* $SO's \implies$ *same St-Traces*

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Endoscopic transfer: spectral side

f. comments

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- **(i)** in the elliptic case the chosen $\Delta(\pi_1, \pi)$ are the only possible coefficients for a spectral version of dual transfer ..., plus they have correct properties re translation principle and parabolic induction ... again this depends also on properties of the geometric factors and compatibility factors

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- **alternate argument** to prove tempered spectral transfer:
- **(i)** in the elliptic case the chosen $\Delta(\pi_1, \pi)$ are the only possible coefficients for a spectral version of dual transfer ..., plus they have correct properties re translation principle and parabolic induction ... again this depends also on properties of the geometric factors and compatibility factors
- **(ii)** theorem is true for some choice of coefficients [old result] and so it is true with the factors $\Delta(\pi_1, \pi)$ we have defined

- still in standard setting, nontempered examples?
  define $\Delta(\pi_1, \pi)$ for very regular pairs in general:
  enough to define $\Delta(\pi_1, \pi; \pi_1', \pi')$ for some tempered (π_1', π'),
  then $\Delta(\pi_1, \pi) := \Delta(\pi_1, \pi; \pi_1', \pi').\Delta(\pi_1', \pi')$

Endoscopic transfer: spectral side

g. standard setting: very regular pairs in general

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- start with elliptic case: construct (π_1', π') tempered elliptic
  or just π_1' tempered elliptic in some cases

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- [remove elliptic assumption]

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- approach to defining tempered spectral factors: again elliptic setting first, translation, and then parabolic descent [Mezo 2013: use results of Duflo for parabolic induction]

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- again similar approach to standard case to define twisted factors $\Delta(\pi_1, \pi)$ for nontempered very regular pairs (π_1, π) ...

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- various (Galois-cohomological) properties of pairings have consequences for harmonic analysis, *e.g.* inversion of spectral transfer in tempered setting

  [Whittaker normalizations $\Longrightarrow$ simplest spectral pairings]

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- now for π_1 take any $\pi^s \in \Pi^s$ = packet attached to φ^s

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- simpler case... **Theorem:** G of quasi-split type, Whittaker norm of absolute Δ, π_0 generic, trivial character $s_{sc} \to \langle \pi_0, s_{sc} \rangle$:
  $\langle \pi, s \rangle := \Delta(\pi^s, \pi)$ gives perfect pairing ... Π as dual of $\mathbb{S}^{ad}$

- **Corollary:** invert transfer in Whittaker setting simply as

$$Trace\ \pi(f) = \left|\mathbb{S}^{ad}\right|^{-1} \sum_{s\in\mathbb{S}^{ad}} \langle \pi, s\rangle\ St\text{-}Trace\ \pi^s(f_1^s) \qquad (11)$$

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- elliptic case, Whittaker setting: calculate $\langle \pi, s \rangle$?
  G^* cuspidal, T anisotropic mod center, also $T_G \subseteq G$
  π = discrete series, π_0 determines Weyl chamber(s) $\mathcal{C}_0$ yielding toral data for T in G^* and then well-defined character κ on $H^1(\Gamma, T^{sc})$; π determines chamber for T_G; inner twist carries this chamber to $\mathcal{C}_0$ up to inner automorphism; make a well defined element ω in $H^1(\Gamma, T^{sc})$; finally, $\langle \pi, s \rangle = \kappa(\omega)$

Structure on packets

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- assume θ preserves fnd. splitting spl_f; may assume inner twist η
  transports spl_f to fnd. splitting spl_f^* of G^* preserved by θ^*,
  spl_f^* provides toral data to transport objects from $G^\vee$...

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- for nontrivial twisting character ϖ, analysis exploits map on endo data: $\mathfrak{e}_z \longrightarrow (\mathfrak{e}_z)_{ad}$ dual to $G_{sc} \longrightarrow G$
  ■